University of Innsbruck

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Loukas Balafoutas, E. Glenn Dutcher,
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Loukas Balafoutas[†], E. Glenn Dutcher[‡], Florian Lindner[†]
and Dmitry Ryvkin[§]

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Abstract

Tournaments are widely used in organizations, explicitly or implicitly, to reward the best-performing employees, e.g., by promotion or bonuses, and/or to penalize the worst-performing employees, e.g., by demotion, withholding bonuses or unfavorable job assignments. These incentive schemes can be interpreted as various *prize allocations* based on the employees' relative performance. While the optimal prize allocation in tournaments of symmetric agents is relatively well-understood, little is known about the impact of the allocation of prizes on the effectiveness of tournament incentive schemes for heterogeneous agents. We show that while multiple prize allocation rules are equivalent when agents are symmetric in their ability, the equivalence is broken in the presence of heterogeneity. Under a wide range of conditions, *loser-prize tournaments*, i.e., tournaments that award a low prize to relatively few bottom performers, are optimal for the firm. The reason is that low-ability agents are discouraged less in such tournaments, as compared to *winner-prize tournaments* awarding a high prize to few top performers, and hence can be compensated less to meet their participation constraints.

JEL classification codes: M52, J33, J24
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[†]University of Innsbruck.
[‡]University of Central Missouri.
[§]Corresponding author: Florida State University, Tallahassee, FL 32306-2180, e-mail: dryvkin@fsu.edu, tel.: +1-850-644-7209, fax: +1-850-644-4535.

1 Introduction

Tournaments, or incentive schemes based on relative performance evaluation, are one of the mainstays in a manager's toolkit of motivational devices. In the workplace, employees may compete with one another to receive a reward, for example in the form of a promotion or bonus (see, e.g., Lazear and Rosen 1981; Bull et al. 1988; Orrison et al. 2004). A sometimes overlooked, but equally important type of workplace tournaments is the competition among co-workers to avoid being penalized. For example, a manager may take an employee off attractive special projects, assign him or her to a more onerous job, or refuse to give an employee an otherwise expected bonus or promotion.[1]

Given the wide use of rank-based rewards and penalties in organizations,[2] it is important to understand what combination of rewards and penalties is optimal for the firm. This question can be formulated quite generally as a *prize allocation* problem. In this paper, we explore how the allocation of prizes alters the effectiveness of tournament contracts. Our model builds on the seminal theory of Lazear and Rosen (1981). Workers perform by choosing effort, which is not observable by the manager. Each worker's performance depends positively on effort but also includes a random component ("noise"). The manager can only observe the ranking of workers by their performance levels and has to design a *tournament contract* that awards fixed prizes based on the workers' ranks. In the baseline case of homogeneous risk-neutral workers, multiple distributions of prizes are efficient and profit-equivalent (Lazear and Rosen 1981), i.e., the predicted work effort and firm profits are the same under those incentive schemes. In this paper, we depart from this symmetric setting and consider heterogeneous workers. Understanding the structure of optimal contracts with heterogeneous agents is important, not only because it may lead to different performance relative to the symmetric setting, but also because of the natural occurrence of heterogeneity within most organizations. We focus on the case of relatively weak heterogeneity because, first, it is analytically tractable, and, second, it is the most relevant for applications due to endogenous labor market sorting and efficiency considerations.[3]

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[1] The most severe penalty is, of course, employee termination. Jack Welch, the former CEO of General Electric, regularly terminated the lowest 10% of the GE employees on the work performance scale. We do not study termination directly in this paper as doing so would require a dynamic model.

[2] For example, a Wall Street Journal article "'Rank and Yank' Retains Vocal Fans" (January 31, 2012, available at `http://online.wsj.com/article/SB10001424052970203363504577186970064375222.html`) states that 60% of Fortune 500 companies currently use some kind of a ranking system for incentive provision.

[3] It is well-established that tournament incentive schemes become increasingly inefficient as the degree of worker heterogeneity rises, due to the discouragement of low-ability workers (see, e.g., O'Keeffe et al. 1984). Thus, tournament contracts are most likely to be used in relatively homogeneous groups.

We show that in the presence of weak heterogeneity the multiplicity of optimal prize allocations is broken in favor of a unique optimal tournament contract. The optimal contract, to the first order in the level of heterogeneity, is a *j-tournament* awarding two distinct prizes:[4] A higher prize is awarded to the agents ranked 1 through j, and a lower prize to the remaining agents. Moreover, in a wide range of cases the optimal contract is a *loser-prize tournament* that awards a low prize to relatively few agents ($j > n/2$, where n is the total number of agents). This result is a consequence of the finding that lower-ability agents are discouraged more in *winner-prize tournaments* that award few high prizes to top performers ($j < n/2$) than in loser-prize tournaments ($j > n/2$), as it is more important for them to avoid losing in the latter. Hence, lower total compensation is needed in loser-prize tournaments to satisfy the agents' participation constraints. This finding may explain the continued widespread use of penalties in firms and implies that those companies that can correctly utilize tournament contracts singling out the worst performers will enjoy an advantage over their competitors.

We also show that tournament contracts for weakly heterogeneous agents are nearly socially efficient, as compared to the fully efficient contracts for symmetric agents. While there is a first-order negative effect of heterogeneity on the firm's profit, it is exactly compensated by a first-order increase in the agents' payoffs. The reduction in social surplus relative to the case of symmetric agents is a much smaller, second-order effect.

We restrict attention to tournament contracts satisfying anonymity, i.e., the principle that two agents cannot be compensated differently for the same output.[5] Such schemes are preferable from a managerial perspective because they do not involve worker discrimination, do not violate procedural equity, and are less demanding in terms of the information the principal needs to possess. As we show, in order to implement an anonymous tournament contract for weakly heterogeneous agents, the principal only needs to know average ability and the ability (but not the identity) of the least productive agent. Finally, as mentioned above, as long as the agents' heterogeneity is not too strong, the inefficiency of anonymous contracts is negligible.

To the best of our knowledge this is the first paper that presents a general, yet tractable, theory of optimal prize allocation for heterogeneous workers in the Lazear and Rosen (1981) framework. In the analysis, we sacrifice precision for generality by using

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[4]We borrow the term "*j*-tournament" from Akerlof and Holden (2012). Importantly, Akerlof and Holden (2012) focus on homogeneous agents and analyze two-prize tournaments in an *ad hoc* fashion, whereas we show that they emerge as a unique optimal mechanism for weakly heterogeneous agents.

[5]It was shown previously that the inefficiencies arising in the traditional tournament contracts in the presence of heterogeneity can be removed by extending the class of possible contracts to those violating the principle of anonymity. Examples of such solutions include ability-specific piece rates (Lazear and Rosen 1981), handicaps (O'Keeffe et al. 1984), and ability-specific prizes (Gürtler and Kräkel 2010).

the linear approximation. This technique is reliable as long as the degree of workers' heterogeneity is not too strong and has proved fruitful in other settings (see, e.g., Fibich and Gavious 2003; Fibich et al. 2004, 2006; Ryvkin 2007, 2009). We show with an example of an otherwise intractable model that the linear approximation agrees with a high-precision numerical solution very well in a wide range of parameters.

The rest of the paper is organized as follows. Section 2 reviews the relevant theoretical and empirical literature on the problem of prize allocation in tournaments. In Section 3, we describe the model and briefly characterize symmetric optimal contracts that are well-known and serve as the point of departure for further analysis. In Section 4, we present the main results and a numerical illustration. Section 5 concludes with a summary and discussion of our findings and their implications.

2 Review of the relevant literature

There is an extensive literature on tournaments in organizations (for a review of the earlier literature see, e.g., McLaughlin 1988, Lazear 1995, Prendergast 1999; for a more recent review see, e.g., DeVaro 2006, Konrad 2009). Most of this literature focuses on tournaments that reward the best-performing employees.[6] Incentive schemes with penalties were initially mentioned by Mirrlees (1975) and later re-examined by Nalebuff and Stiglitz (1983), who note the equivalence of multiple prize allocation schemes in the symmetric case.[7]

Most of the existing theoretical literature on optimal prize allocation in tournaments focuses on two classes of models – perfectly discriminating contests and Tullock (1980) contests (for a detailed review, see Sisak 2009). Moldovanu and Sela (2001) study perfectly discriminating contests that are essentially all-pay auctions with private and possibly non-linear bidding costs. They find that the optimal allocation of prizes that maximizes total effort depends on the curvature of the effort cost function: One top prize is optimal for linear or concave costs, while multiple prizes can be optimal for convex costs. Moldovanu

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[6]Throughout this discussion, we focus on the standard static principal-agent models of tournaments in the tradition of Lazear and Rosen (1981). There is also an extensive literature on dynamic tournaments involving sequential elimination of employees (see, e.g., Rosen 1986, O'Flaherty and Siow 1995, Gradstein and Konrad 1999, Ryvkin and Ortmann 2008, Casas-Arce and Martinez-Jerez 2009, Fu and Lu 2009). Although elimination can be thought of as a form of penalty, it is typically not discussed as such. Instead, these models focus on the incentives of the remaining (promoted) agents.

[7]The equivalence of optimal tournament contracts with various configurations of prizes in the symmetric case was mentioned already by Lazear and Rosen (1981). Nalebuff and Stiglitz (1983) discuss the equilibrium existence and note that loser-prize tournament structures tend to reduce nonconvexities in the principal-agent problem: In loser-prize tournaments the agents' payoff functions remain concave as the number of agents n increases, whereas the pure strategy equilibrium disappears as n increases in winner-prize tournaments.

et al. (2012) explore optimal prize structures in the same framework but explicitly allow for punishment (negative prizes) which may or may not be costly to the employer. They identify the relationship between the distribution of ability in the population and the prize structure and show that, in some cases, punishment can be optimal even if it is costly. Baye et al. (1996), Barut and Kovenock (1998) and Clark and Riis (1998), among others, study all-pay auctions of complete information and also find that multiple prizes can be optimal for some configurations of types. In the Tullock (1980) framework, it was found that heterogeneity (Baik 1994, Szymanski and Valetti 2005) and a sufficiently high discriminatory power (Blavatskyy 2004) can lead to the optimality of the second prize. Schweinzer and Segev (2012) show that multiple prizes can be optimal also in symmetric Tullock contests with a nested winner determination structure. Liu et al. (2013) study optimal prize allocation in tournaments as a general mechanism design problem under incomplete information and show that punishments arise as part of a second-best solution.

The paper that is related most closely to ours is by Akerlof and Holden (2012; henceforth, AH12) who study optimal prize structures using the Lazear and Rosen (1981) framework.[8] They consider homogeneous agents and focus on the role of the shape of agents' utility function (risk aversion and prudence) in determining the optimal prize structure. They find that nontrivial profiles of prizes involving more than two distinct prizes can be optimal depending on parameters. Our paper can be viewed as complementary to AH12 as we use a model with risk-neutral agents but focus on the effect of agents' heterogeneity in ability. In the extended working paper version of AH12, Akerlof and Holden (2007) provide some results for heterogeneous agents. First, they discuss a model in which agents learn their abilities after they choose effort levels; thus, agents are symmetric *ex ante* but heterogeneous *ex post*. The resulting equilibrium is symmetric and has properties similar to the equilibrium with *ex ante* symmetric agents. Second, Akerlof and Holden (2007) discuss some special cases of models with *ex ante* heterogeneous agents, restricting attention to tournaments with only two types of agents and an equal number of agents of each type ($n/2$ high ability agents and $n/2$ low ability agents); they also restrict the shape of the effort cost function to quadratic (in the case of additive heterogeneity) or power law (in the case of multiplicative heterogeneity). For additive heterogeneity, Akerlof and Holden (2007) show that the tournament that pays a low prize w_2 to the lowest-ranked agent and a high prize w_1 to the remaining $n-1$ agents (the "strict loser-prize tournament") induces a higher level of effort than the tournament that pays prize

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[8]Krishna and Morgan (1998) pose essentially the same question but restrict attention to tournaments of up to four agents.

w_1 to the highest-ranked agent and prize w_2 to the remaining $n-1$ agents (the "strict winner-prize tournament"). The applicability of this exercise may be limited, however, because total compensation is clearly higher in the former tournament than in the latter, and hence the two incentive schemes are not directly comparable. For multiplicative heterogeneity, Akerlof and Holden (2007) show that when heterogeneity is sufficiently large, the strict winner-prize scheme is preferred to the strict loser-prize scheme. In contrast to Akerlof and Holden (2007), our model does not restrict the number of player types, nor does it impose any parametric restrictions on the cost function of effort. Additionally, we keep various prize structures comparable by calculating optimal contracts in all cases.

Kräkel (2000) discusses tournaments in which workers may face "relative deprivation," a behavioral term in the payoff function making a worker minimize the distance between her income and the average income of a richer reference group. One of the results is that in the absence of relative deprivation, for symmetric workers, strict winner-prize tournaments are more effective than strict loser-prize tournaments from the organizer's perspective. An important difference between our approach and that of Kräkel (2000) is that he does not calculate optimal contracts, and the result is driven by the assumption that the high and low prizes are the same in both tournament schemes and thus the strict loser-prize tournament always costs more to the organizer.

Gürtler and Kräkel (2012) study rank-order "dismissal tournaments" of two heterogeneous workers, one of whom is terminated as a result. Their primary focus is on the selection efficiency of the termination mechanism, defined as the probability that the high-ability worker is retained. Gürtler and Kräkel (2012) show that, if the low-ability worker has a relatively low outside option, potential termination incentivizes her more than the high-ability worker. This leads to the possibility that, in some instances, the high-ability worker contributes less effort and is more likely to be terminated. Kräkel (2012) uses a similar argument to discuss adverse selection in a sequential elimination setting.

3 The model

3.1 Model setup

Consider a tournament of $n \geq 2$ risk-neutral agents indexed by $i = 1, \ldots, n$. Each agent participates in the tournament by exerting effort $e_i \geq 0$ that costs her $c_i g(e_i)$. Here, $c_i > 0$ is the agent's cost parameter (higher c_i implies lower ability), and $g(\cdot)$ is a strictly convex and strictly increasing function, with $g(0) = 0$. All agents have the same outside option

payoff ω.[9]

Following Lazear and Rosen (1981), we model agent i's output as $y_i = e_i + u_i$, where u_i is a zero-mean random shock. Shocks $u_1, \ldots, u_n$ are independent across individuals and drawn from the same distribution with cumulative density function (cdf) $F(u)$ and probability density function (pdf) $f(u)$.[10] Let $u_l < 0$ and $u_h > 0$ denote, respectively, the lower and upper bounds of the support of F, which may be finite or infinite.

The agents' output levels are ranked, and the agent ranked r receives prize V_r, with $V_1 \geq V_2 \geq \ldots \geq V_n$, where at least two prizes are distinct.[11] Let $p^{(i,r)}(\mathbf{e})$ denote the probability, as a function of the vector of effort levels $\mathbf{e} = (e_1, \ldots, e_n)$, that agent i's output is ranked r in the tournament. Agent i's expected payoff then can be written as

$$\pi_i(\mathbf{e}) = \sum_{r=1}^{n} p^{(i,r)}(\mathbf{e})V_r - c_i g(e_i).$$

For a given configuration of prizes, suppose an equilibrium in pure strategies exists and let $\mathbf{e}^* = (e_1^*, \ldots, e_n^*)$ denote the vector of equilibrium effort levels. There is a risk-neutral principal, whose objective function is the expected profit defined as the difference between aggregate effort and total prize payments, $\Pi = \sum_i e_i - \sum_r V_r$.[12] The principal chooses a tournament contract $(V_1, \ldots, V_n)$. Given the principal's objective, the optimal contract $(V_1^*, \ldots, V_n^*)$ solves

$$\max_{V_1, \ldots, V_n} \sum_i e_i^* - \sum_r V_r$$

subject to the participation constraints, $\pi_i(\mathbf{e}^*) \geq \omega$, $i = 1, \ldots, n$, and the incentive compatibility constraints ensuring that $\mathbf{e}^*$ is an equilibrium under the optimal contract.[13]

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[9]Outside option ω is the expected payoff of an agent if she does not participate in the tournament. It can represent unemployment insurance benefits, earnings in a different firm or sector, or income from self-employment. The assumption that ω is homogeneous across agents is warranted provided their abilities are part of job-specific human capital and thus not transferable outside the firm. If it is not the case, agents' outside options can be correlated with abilities (see, e.g., Kräkel 2012). We discuss implications of such a correlation in Section 4.4.

[10]Under risk-neutrality, the results do not change if shocks u_i contain an additive common shock component, i.e., $u_i = \rho + \epsilon_i$ where ρ is the common shock and ϵ_i are zero-mean i.i.d.

[11]Similar to Moldovanu and Sela (2001), we restrict attention to monotone prize schedules. Even though nonmonotone prize schedules can be optimal for symmetric agents (cf. AH12), they are difficult to rationalize in a realistic organizational setting.

[12]The results below are also valid for a more general model with $\Pi = Q(\sum_i e_i) - \sum_r V_r$, where $Q(\cdot)$ is a smooth, strictly increasing and concave function. The results below correspond to normalization $Q'(n\bar{e}^s) = 1$, which can be adopted without loss of generality.

[13]Thus, similar to AH12, we assume that the firm is a profit-maximizing monopolist. This setting is different from that of Lazear and Rosen (1981) who assume that the firm operates in a competitive market under the zero profit condition and maximizes workers' payoffs. For symmetric agents, both settings lead to the same (socially optimal) level of effort and profiles of optimal prizes that differ only by an additive constant. That is, there is a constant T such that if $(V_1, \ldots, V_n)$ is optimal in one setting

3.2 Symmetric optimal contracts

The results of this section are well-known in the literature. We provide them here for completeness because they serve as the point of departure for the analysis that follows. Assume that all agents have the same ability, $c_1 = \ldots = c_n = \bar{c}$. In this section, we briefly characterize the symmetric equilibrium assuming it exists. The equilibrium existence/uniqueness issue for this class of games has not been resolved to date, and is outside of the scope of this paper.[14]

Let $\bar{e}$ denote the symmetric equilibrium effort level. For a given configuration of prizes, $\bar{e}$ solves the symmetrized first-order condition

$$\sum_r \beta_r V_r = \bar{c} g'(\bar{e}), \tag{1}$$

where $\beta_r = p_1^{(1,r)}(\bar{e}, \ldots, \bar{e})$ is the derivative of an agent's probability to be ranked r with respect to the agent's own effort evaluated at the symmetric equilibrium point. The expression for β_r is provided in AH12:

$$\beta_r = \binom{n-1}{r-1} \int F(t)^{n-r-1}[1 - F(t)]^{r-2}[n - r - (n-1)F(t)]f(t)^2 dt. \tag{2}$$

Coefficients β_r, referred to by AH12 as "weights," play a critical role in determining the optimal distribution of prizes for symmetric risk-averse agents. As we show below, however, a different set of coefficients enters the stage for heterogeneous agents.

Weights β_r are determined entirely by the distribution of noise F. The following

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then $(V_1 + T, \ldots, V_n + T)$ is optimal in the other. However, for heterogeneous agents the situation is more complicated. In our (monopolistic) setting, the optimal contract maximizes the firm's profit subject to the participation constraint for the lowest ability agent. This leads to a contract that, effectively, maximizes that agents' payoff. For the Lazear-Rosen (competitive) setting to lead to equivalent results (where equivalence is understood in the same sense as for the case of symmetric agents), it would have to be assumed that the firm's profit is zero and the lowest-ability agent's payoff is maximized. However, such maximization could lead to lower payoffs for high-ability agents who, in a competitive setting, then may choose to go to another firm that gives them a better deal, albeit at the expense of losing low-ability agents. In our setting, this is not a problem because the firm is a monopolist and those agents' participation constraints would be satisfied automatically. For the equivalence of the two settings to hold, only certain types of movements of workers between firms would have to be allowed in the competitive setting. Specifically, one would need to assume that the number and ability profiles of workers must be the same across firms. That is, firms need all n worker types and compete in a separate market for each type. The formulation with a monopolistic firm allows us to avoid these issues.

[14]We focus on the symmetric equilibrium as the point of departure, even though it may not be the only possible equilibrium, because it is the most "natural" equilibrium for symmetric agents. It is generally understood that a symmetric pure strategy equilibrium exists for a sufficiently large variance of noise and sufficiently convex effort cost function $g(\cdot)$, cf. the discussion by Nalebuff and Stiglitz (1983). AH12 discuss second-order conditions in a more general setting with risk-averse agents, but still they do not provide a sufficient condition for a *global* maximum of the symmetrized payoff function.

additional properties of β_r are provided by AH12: (i) For any distribution F, $\sum_r \beta_r = 0$, $\beta_1 \geq 0$, and $\beta_n \leq 0$; (ii) If F is symmetric, i.e., $f(t) = f(-t)$, then $\beta_r = -\beta_{n-r+1}$ for all r; (iii) If F is a uniform distribution on the interval $[-b, b]$, then $\beta_1 = -\beta_n = 1/(2b)$ and $\beta_r = 0$ for $1 < r < n$.

A critical issue that arises in the analysis below, and is also discussed by AH12, is whether weights β_r are monotonically decreasing in r. Although this appears to be the case for some prominent distributions (such as the uniform and the normal distributions), the monotonicity of β_r is not a universal property. Specifically, as mentioned by AH12, nonmonotonicities in the weights tend to arise when F is multimodal. In what follows, we will be making the assumptions of monotonicity of β_r and/or symmetry of F whenever necessary.

In the symmetric equilibrium, the probability of any agent winning the tournament is $1/n$; therefore, the equilibrium payoff of an agent is $\bar{\pi} = (1/n) \sum_r V_r - \bar{c}g(\bar{e})$. To calculate the optimal contract, write the principal's profit as $\bar{\Pi} = n[\bar{e} - (1/n) \sum_r V_r]$. Effort is costly, and compensation is independent of effort; therefore, the participation constraint binds, $\bar{\pi} = \omega$. This gives $\bar{\Pi} = n[\bar{e} - \bar{c}g(\bar{e}) - \omega]$. The principal will choose an optimal contract $(\bar{V}_1, \ldots, \bar{V}_n)$ such that the equilibrium effort $\bar{e}$ maximizes $\bar{\Pi}$. This gives the following system of equations:

$$\sum_r \beta_r V_r = \bar{c}g'(\bar{e}), \quad \sum_r V_r = n[\omega + \bar{c}g(\bar{e})], \quad \bar{c}g'(\bar{e}) = 1. \tag{3}$$

Let $\bar{e}^s$ denote the solution of the equation $\bar{c}g'(\bar{e}) = 1$. Then any configuration of prizes $(\bar{V}_1, \ldots, \bar{V}_n)$ that solves the system of equations

$$\sum_r \beta_r V_r = 1, \quad \sum_r V_r = n[\omega + \bar{c}g(\bar{e}^s)]. \tag{4}$$

will implement an optimal contract. The firm's optimal profit is $\bar{\Pi}^s = n[\bar{e}^s - \bar{c}g(\bar{e}^s) - \omega]$. The resulting contracts are socially optimal, in the sense that they maximize total surplus $n[\bar{e} - \bar{c}g(\bar{e})]$.

In what follows, we will assume that $\bar{\Pi}^s$ is strictly positive, i.e., that $\bar{e}^s - \bar{c}g(\bar{e}^s) - \omega > 0$. First, it is reasonable for a monopolistic firm to earn a positive profit. Second, the firm's profit will necessarily fall below $\bar{\Pi}^s$ when agents are heterogeneous, and this assumption enables the firm to make a positive profit even then and ensures that, for weakly heterogeneous agents, the firm will prefer to implement an equilibrium with all n agents participating. For a detailed discussion, see Appendix A.4.

As seen from Eqs. (4), an optimal contract is only determined up to $n - 2$ arbitrary

prizes. Thus, two distinct prizes are sufficient to generate an optimal contract. The multiplicity of optimal contracts, the discussion of which goes back to Lazear and Rosen (1981), is a consequence of the symmetry (and risk-neutrality) of agents. As we show below, the multiplicity of optimal contracts will be broken when agents are heterogeneous, and in a wide range of scenarios a unique optimal contract will emerge.

4   Optimal contracts with weakly heterogeneous agents

4.1   Equilibrium with weakly heterogeneous agents

We now turn to tournaments of heterogeneous agents. While the case of arbitrary heterogeneity is analytically intractable, a lot can be said about the impact of relatively weak heterogeneity. From a practical viewpoint, weak heterogeneity means that agents' abilities are not very different from some average level. This is a reasonable assumption to make in most cases, as employees whose abilities are substantially different from group average are unlikely to be part of a tournament in the first place, due to the well-documented adverse effects of agent disparity on tournament efficiency (e.g., Lazear and Rosen 1981, O'Keeffe et al. 1984, Müller and Schotter 2010). For example, the organization can benefit by splitting employees into cohorts by ability and conducting separate parallel tournaments within those cohorts. Moreover, in many cases natural job market sorting will lead to attrition of employees whose ability is too far from the firm's average.

Let $\bar{c} = n^{-1} \sum_i c_i$ denote the average cost parameter. Introduce relative abilities (or, for brevity, abilities) a_i defined as negative relative deviations of cost parameters from the average: $c_i = \bar{c}(1 - a_i)$. By construction, $a_i < 1$, $\sum_i a_i = 0$, and higher a_i implies lower cost of effort, i.e., a higher ability. Moreover, $a_i > 0$ ($a_i < 0$) implies ability above (below) average.

Assume agents are weakly heterogeneous, in the sense that $\mu \equiv \max_i |a_i| \ll 1$. Thus, it is assumed that relative deviations of cost parameters c_i from the average cost parameter $\bar{c}$ are "small."

In what follows, we will assume that, for a given configuration of prizes $(V_1, \ldots, V_n)$, the pure strategy equilibrium with weakly heterogeneous agents exists and is governed by the corresponding system of first-order conditions:

$$\sum_r p_i^{(i,r)}(\mathbf{e})V_r = c_i g'(e_i), \quad i = 1, \ldots, n. \tag{5}$$

This is a reasonable assumption to make provided the symmetric equilibrium exists and the agents' payoffs are smooth functions of parameters in the neighborhood of the symmetric equilibrium point.[15] In this case, we can look for the equilibrium effort levels in the form $e_i = \bar{e}(1 + x_i)$, where the relative deviations of effort from the symmetric equilibrium level, x_i, are also "small," $|x_i| \ll 1$. In the linear approximation, x_i can be found approximately, with accuracy $O(\mu^2)$, by expanding the first-order conditions (5) around the symmetric equilibrium point to the first order in μ. The result is given by the following proposition (all proofs are provided in the Appendix).

Proposition 1 *For a given configuration of prizes $(V_1, \ldots, V_n)$, in the linear approximation,*

(a) the equilibrium effort of agent i is $e_i^* = \bar{e}(1 + x_i)$, with

$$x_i = \xi(\bar{e})a_i + O(\mu^2), \quad \xi(\bar{e}) = \frac{\bar{c}g'(\bar{e})}{\bar{e}[\bar{c}g''(\bar{e}) - \sum_r \lambda_r V_r]}; \tag{6}$$

(b) the equilibrium payoff of agent i is

$$\pi_i = \frac{1}{n}\sum_r V_r - \bar{c}g(\bar{e}) + \eta(\bar{e})a_i + O(\mu^2), \quad \eta(\bar{e}) = \frac{\bar{c}^2 g'(\bar{e})^2}{(n-1)[\bar{c}g''(\bar{e}) - \sum_r \lambda_r V_r]} + \bar{c}g(\bar{e}). \tag{7}$$

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[15]For heterogeneous players, the conditions for equilibrium existence become even more stringent, but they still work along the same lines as in the case of symmetric agents. Think of the system of first-order conditions (5) as defining implicitly functions $e_i^*(c_1, \ldots, c_n)$. Equation (6) in Proposition 1 below gives the implicit derivatives $\partial e_i^*/\partial c_j$ evaluated at the symmetric equilibrium point $\bar{e}$, $\bar{c}$. As long as these derivatives exist, by the implicit function theorem there is a neighborhood around $\bar{c}$ where the system of first-order conditions (5) has a solution. Thus, the question of equilibrium existence reduces to whether in that neighborhood the first-order conditions are *sufficient* conditions for global best responses, i.e., whether payoff functions $\pi_i(\mathbf{e}) = \sum_r V_r p^{(i,r)}(\mathbf{e}) - c_i g(e_i)$ are quasi-concave in e_i for each i. This can be achieved, similar to the case of symmetric agents, if costs $c_i g(e_i)$ are sufficiently convex and $p^{(i,r)}(\mathbf{e})$ is not "too convex" in e_i, that is, the marginal return to effort is not too high, i.e., the variance of noise is large enough. The approximate equilibrium with weakly heterogeneous agents is unique in the neighborhood of the symmetric equilibrium by construction, as it is given by the solution to a system of linear equations with full rank.

Here,

$$\lambda_r = \Delta_r \tag{8}$$
$$+ \frac{n(n-2)![(r-1)(r-2)M_r - 2(r-1)(n-r)M_{r+1} + (n-r)(n-r-1)M_{r+2}]}{2(n-r)!(r-1)!},$$

$$M_k = \int F(t)^{n-k}[1-F(t)]^{k-3}f(t)^3 dt, \tag{9}$$

$$\Delta_r = \begin{cases} f(u_h)^2 - f(u_l)^2, & n=2, \ r=1 \\ f(u_l)^2 - f(u_h)^2, & n=2, \ r=2 \\ \frac{nf(u_h)^2}{2}, & n \geq 3, \ r=1 \\ -\frac{n}{2}[f(u_l)^2 I_{n=3} + f(u_h)^2], & n \geq 3, \ r=2 \\ -\frac{n}{2}[f(u_l)^2 + f(u_h)^2 I_{n=3}], & n \geq 3, \ r=n-1 \\ \frac{nf(u_l)^2}{2}, & n \geq 3, \ r=n \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

$I_{n=3}$ is the indicator equal 1 if $n = 3$ and zero otherwise.

In what follows, we will assume that the denominator in the expression for $\xi(\bar{e})$, Eq. (6), is positive, i.e., higher ability agents exert higher effort, as would be expected in a "well-behaved" equilibrium.[16]

Proposition 1 shows that the deviations of agents' efforts and payoffs from the symmetric equilibrium levels are determined, in the linear approximation, by coefficients λ_r. As we show below, these coefficients, together with β_r, also determine the prize structure of optimal contracts.

Note that for $n = 2$, $\lambda_r = \Delta_r$; moreover, if F is symmetric, $\lambda_r = 0$. The following corollary follows directly from Eq. (8) and describes the properties of coefficients λ_r for $n \geq 3$.

Corollary 1 *For $n \geq 3$, coefficients λ_r have the following properties:*
(i) For any distribution F, $\sum_r \lambda_r = 0$, $\lambda_1 \geq 0$ and $\lambda_n \geq 0$;
(ii) If F is symmetric, $\lambda_r = \lambda_{n-r+1}$ for all r;
(iii) If F is a uniform distribution on the interval $[-b, b]$ then $\lambda_r = \Delta_r$, i.e., $\lambda_1 = \lambda_n = n/8b^2$; $\lambda_2 = \lambda_{n-1} = -n(I_{n=3}+1)/8b^2$; and $\lambda_r = 0$ for $2 < r < n-1$.

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[16]This is, of course, a consequence of our specification of the cost of effort in which effort and ability are complementary.

4.2 Optimal contracts

It follows from Proposition 1, Eq. (6), that the aggregate deviation of agents' effort from the symmetric equilibrium level is zero in the linear approximation, $\sum_i x_i = O(\mu^2)$; therefore, in the linear approximation the aggregate effort of agents is the same as in the symmetric tournament and, thus, the principal's objective function is $\Pi = n\bar{e} - \sum_r V_r + O(\mu^2)$. The first-order correction to the principal's profit arises due to the participation constraint that now will be binding only for the lowest-ability agent. Let agents be ordered, without loss of generality, so that $c_1 \leq c_2 \leq \ldots \leq c_n$. Then the participation constraint will be $\pi_n = \omega$, where π_n is the equilibrium payoff of agent n given by Eq. (7). This gives the principal's objective function $\Pi = n[\bar{e} - \bar{c}g(\bar{e}) - \omega + \eta(\bar{e})a_n] + O(\mu^2)$. The principal will choose the optimal contract $(V_1^*, \ldots, V_n^*)$ such that the average equilibrium effort $\bar{e}$ maximizes Π and satisfies the participation constraint $(1/n)\sum_r V_r - \bar{c}g(\bar{e}) + \eta(\bar{e})a_n = \omega$.

Note first that, in the linear approximation, the principal can implement average equilibrium effort $\bar{e} = \bar{e}^s$. The reason is that, due to the envelope theorem, there is no first-order effect of a deviation of $\bar{e}$ from $\bar{e}^s$ on the principal's profit Π. Interestingly, this also implies that the principal can implement *any* average equilibrium effort close to $\bar{e}^s$, $\bar{e} = \bar{e}^s + \epsilon$, where $\epsilon = O(\mu)$, and obtain the same profit, in the linear approximation. Whichever effort the principal implements does not change the structure or efficiency of optimal contracts; therefore, for simplicity, we assume in the remainder of this section that the principal implements average effort $\bar{e}^s$. The more general case is presented in the Appendix.

A *j-tournament*, as defined by AH12, is a tournament prize structure that awards two distinct prizes, a prize W_1 to the agents ranked 1 through j and a prize W_2 to the agents ranked $j+1$ through n, with $W_1 > W_2$. It turns out that, in the linear approximation, the optimal tournament prize structure in the tournament of weakly heterogeneous agents is that of a *j*-tournament. The results are summarized in the following proposition.

Proposition 2 *In the tournament of weakly heterogeneous agents, in the linear approximation:*
(a) The optimal contract is a j-tournament, with $V_1^* = \ldots = V_j^* = W_1$, $V_{j+1}^* = \ldots = V_n^* = W_2$, and

$$j \in \arg\min_{1 \leq r \leq n-1} \frac{\Lambda_r}{B_r}.$$

Here,

$$B_r = \frac{(n-1)!}{(n-r-1)!(r-1)!} \int F(t)^{n-r-1}[1-F(t)]^{r-1}f(t)^2 dt, \tag{11}$$

$$\Lambda_r = \frac{n(n-2)!}{2(n-r-1)!(r-1)!}[(n-r-1)M_{r+2}-(r-1)M_{r+1}] + \sum_{k=1}^{r}\Delta_k.$$

(b) The optimal prizes are

$$W_1 = \omega + \bar{c}g(\bar{e}^s) + \frac{n-j}{nB_j} - \eta(\bar{e}^s)a_n + O(\mu^2), \tag{12}$$

$$W_2 = \omega + \bar{c}g(\bar{e}^s) - \frac{j}{nB_j} - \eta(\bar{e}^s)a_n + O(\mu^2).$$

(c) The firm's optimal profit is

$$\Pi^* = \bar{\Pi}^s + n\eta(\bar{e}^s)a_n + O(\mu^2), \tag{13}$$

The resulting optimal contract is nearly efficient (the inefficiency is of order $O(\mu^2)$). To the first order in μ, heterogeneity leads to a *redistribution* of surplus from the principal to the agents, but not to a reduction in surplus. Indeed, by construction, $a_n < 0$; therefore, in the heterogeneous case the principal's profit is reduced, in the linear approximation, by $n\eta(\bar{e}^s)|a_n|$, as compared to the symmetric case, and the agents' aggregate compensation is increased by the same amount.

Proposition 2 is the central result of this paper. It shows that the multiplicity of optimal contracts with symmetric agents is broken in the presence of weak heterogeneity. Two distinct prizes are still sufficient to implement an optimal contract in the linear approximation, but the structure of the contract is determined critically by the distribution of noise through coefficients β_r and λ_r. Unfortunately, not much can be said about the properties of these coefficients for general distributions F, and thus the optimal j in the j-tournament prize structure can potentially be (almost) anywhere. In the remainder of this section, we will explore how certain restrictions imposed on F lead to restrictions on the location of j.

Following AH12, we will refer to j-tournaments with $j \le n/2$ as "winner-prize tournaments" because they award the high prize to relatively few top performers; and to j-tournaments with $j \ge n/2$ as "loser-prize tournaments" because they award the low prize to relatively few bottom performers. We will also use the terms "strict winner-prize tournament" and "strict loser-prize tournament" to refer to the extreme versions of the

two tournaments with $j = 1$ and $j = n - 1$, respectively.[17] In what follows, we show that for a wide class of distributions F the optimal tournament prize structure with weakly heterogeneous agents is that of a loser-prize tournament.
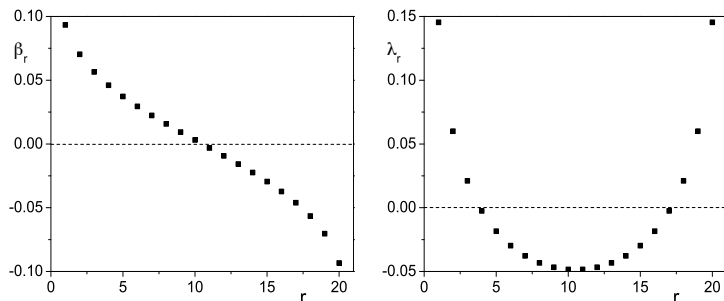

Figure 1: Coefficients β_r (left) and λ_r (right) as functions of r for $n = 20$ and the normal distribution of noise F with zero mean and unit variance.
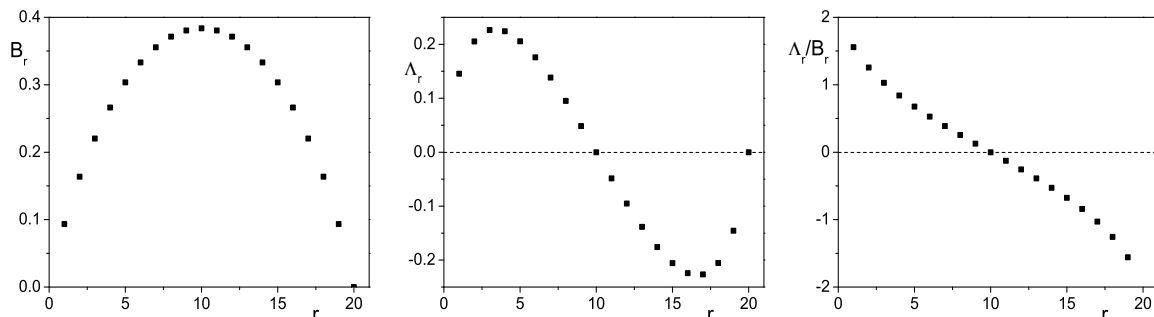

Figure 2: Coefficients B_r (left), Λ_r (center), and their ratio Λ_r/B_r (right) as functions of r for $n = 20$ and the normal distribution of noise F with zero mean and unit variance.

Figure 1 shows β_r and λ_r as functions of r for the normal distribution of noise with $n = 20$. As seen from Figure 1, both coefficients exhibit the predicted symmetry. Moreover, β_r is monotonically decreasing in r, while λ_r is U-shaped. These shapes are quite generic and hold for a variety of single-peaked symmetric distributions. They have consequences for the dependence of the cumulative coefficients, B_r and Λ_r, on r, as shown in Figure 2.

Figure 2 shows the dependence of B_r, Λ_r and their ratio, Λ_r/B_r, on r for the same distribution of noise as in Figure 1. Recall that B_r is positive for any distribution F (cf. Eq. (11)) and will have the inverted-U shape as in Figure 2 (left) if β_r is decreasing in r. The maximum of B_r will be reached at the point where β_r crosses zero. It will be

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[17]Kräkel (2000) referred to the prize structures with $j = 1$ and $j = n - 1$ as "bonus" and "penalty" schemes, respectively. Dutcher et al. (2015) referred to them, respectively, as "winner" and "loser" tournaments.

in the middle if β_r is symmetric (i.e., if distribution F is symmetric). Similarly, recall that $\sum_r \lambda_r = 0$ and λ_1 is positive for any distribution F; therefore, if λ_r is U-shaped as in Figure 1, Λ_r will be positive and will have a maximum at a relatively low r, then it will cross into the negative domain and will have a minimum for a relatively high r, as in Figure 2 (center). It will be symmetric around the middle if F is symmetric.

The ratio Λ_r/B_r appears to be monotonically decreasing in r when F is the normal distribution (Figure 2, left), and reaches its minimum for $r = n - 1$. Thus, when F is the normal distribution, the optimal contract is the strict loser-prize tournament awarding prize W_1 to the agents ranked 1 though $n-1$ and prize W_2 to the agent ranked last. It is easy to see that the same is true when F is a uniform distribution. A more general result is given by the following proposition.

Proposition 3 *(a) Suppose the distribution of noise is symmetric and λ_r is U-shaped. Then the optimal tournament contract for weakly heterogeneous agents, in the linear approximation, is a loser-prize tournament.*
(b) For any distribution F under no circumstances is the strict winner-prize tournament optimal for $n \geq 3$.

To see why Proposition 3 is true, consider the shapes of B_r and Λ_r (Fig. 2). It is clear that the minimum of Λ_r/B_r will be reached when $\Lambda_r < 0$; therefore the optimal j cannot be equal to 1 for any F, and has to be greater than $n/2$ when F is symmetric.

Our results imply that, when agents are weakly heterogeneous, firms that use loser-prize tournaments awarding a low prize to relatively few worst-performing workers will perform better.

4.3   A numerical illustration

In this section, we provide a numerical illustration of the results summarized in Propositions 1 and 2. The goal of this section is to demonstrate that the linear approximation approach used in Sections 4.1 and 4.2 produces results that are very close to high-precision numerical solutions in a wide range of parameters.

For illustration, consider a tournament of $n = 4$ agents with the cost of effort $g(e) = e^2/2$, the standard normal distribution of noise F, and the outside option $\omega = 0$. The average cost parameter $\bar{c} = 1$, and the agents' relative abilities are $a_1 = d$, $a_2 = d/3$, $a_3 = -d/3$ and $a_4 = -d$. Here, $d \geq 0$ is the heterogeneity parameter, with $d = 0$ corresponding to the homogeneous case. The weak heterogeneity approximation requires that d be small compared to unity. For practical purposes, $d \leq 0.1$ would typically be considered as "small" in applied mathematics. As we show below, the linear approximation in this

example works remarkably well at least for $d \leq 0.1$, which corresponds to a nearly 20% variation in ability between the highest and the lowest ability agents.

We start with an illustration of the linearized equilibrium characterized in Proposition 1. Let the prizes be $V_1 = 2$, $V_2 = 1$, $V_3 = 0$ and $V_4 = 0$. This configuration of prizes is not optimal, but we use it here to demonstrate that the linear approximation works well for various configurations of prizes, not necessarily restricted to two-prize optimal contracts described in Proposition 2. The left panel in Figure 3 shows the dependence of equilibrium effort levels e_i^* on the heterogeneity parameter d for each of the four agents.
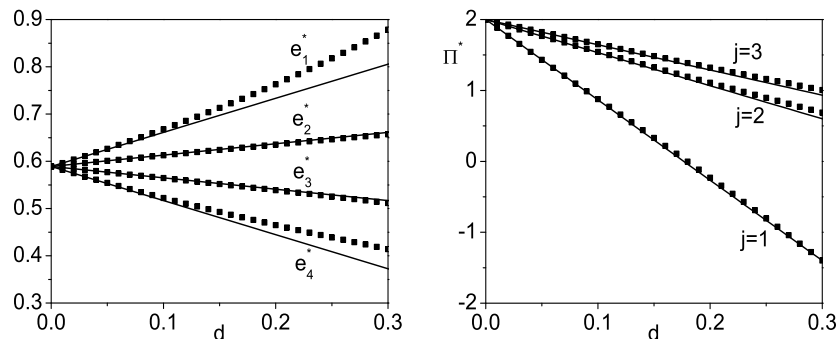

Figure 3: *Left*: Equilibrium efforts e_i^* as functions of heterogeneity parameter d. The solid lines show the linear approximation given by Eq. (6). The squares show the results of a high-precision numerical solution of the system of Eqs. (5). *Right*: The firm's profit Π^* as a function of heterogeneity parameter d for j-tournaments with $j = 1, 2, 3$. The solid lines show the linear approximation given by Eq. (13). The squares show the results of a high-precision numerical computation of the profit in the equilibrium generated by the corresponding j-tournament.

The solid lines in the left panel show the linear approximation $e_i^* = \bar{e}(1 + x_i)$, with $\bar{e} = 0.589$ and x_i given by Eq. (6).[18] The squares show the results of a high-precision numerical solution of the system of Eqs. (5). As expected, the equilibrium efforts are ranked in the same way as relative abilities, with more able agents exerting higher effort. As d increases, variation in effort between agents becomes substantial, and it is captured remarkably well by the linear approximation. In order to quantify the accuracy of the linearized solution, consider the maximal relative deviation of the linearized solution, $e_i^{*,\mathrm{lin}}(d)$, from the high-precision numerical solution, $e_i^{*,\mathrm{num}}(d)$, defined over an interval D of the heterogeneity parameter d:

$$\theta = \max_{d \in D} \max_i \left| \frac{e_i^{*,\mathrm{lin}}(d) - e_i^{*,\mathrm{num}}(d)}{e_i^{*,\mathrm{num}}(d)} \right|. \tag{14}$$

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[18]Recall that $\bar{e}$ is the solution of the equation $\sum_r \beta_r V_r = \bar{c} g'(e)$. In our example with $n = 4$ and the standard normal distribution F, Eq. (2) gives $\beta_1 = 0.257$, $\beta_2 = 0.0743$, $\beta_3 = -0.0743$ and $\beta_4 = -0.257$.

For $D = [0, 0.1]$, we obtain $\theta = 0.0107$; that is, the maximal relative error of the linear approximation for equilibrium efforts is around 1% for $d \leq 0.1$.

We now turn to an illustration of Proposition 2. A complete numerical computation of optimal contracts is prohibitively complex because it requires optimization of the firm's profit Π as a function of prizes $V_1, \ldots, V_4$, with the exact equilibrium computed at each step of the optimization process. We, therefore, present hybrid computational results. Since we already know, from the illustration above, that the equilibrium is evaluated very well by the linear approximation as long as d is not too large, we use the optimal prizes W_1^* and W_2^* computed in the linear approximation, Eq. (12), and calculate the exact profit of the firm, Π^*, for every j-tournament (with $j = 1, 2, 3$) generated by those prizes. The results are presented in the right panel of Figure 3 that shows the firm's optimal profit, Π^*, as a function of d for the three j-tournaments. The linear approximation, Eq. (13), is shown by the solid lines, while the squares show the results of a high-precision computation of Π^*. As seen from the figure, the optimal j-tournament is the strict loser-prize tournament with $j = 3$, as predicted by Proposition 2.[19] As expected, the firm's optimal profit decreases with heterogeneity. The agreement between the numerically computed profit and the linear approximation is excellent. The maximal relative error for payoffs Π^* defined similar to (14) gives $\theta = 0.0136$, i.e., less than 2%, for $d \leq 0.1$.

4.4   Heterogeneous outside options

One possible extension of the analysis presented above is to explore the effect of heterogeneity in the agents' outside options. Given that the agents' heterogeneity in ability is weak, it is reasonable to assume that their outside option payoffs ω_i are also close to the average value ω. Let $\omega_i = \omega + \kappa_i$, where $\sum_i \kappa_i = 0$ and $|\kappa_i/\omega| \ll 1$.

The outside options will affect the principal's problem through the participation constraints that will now take the form $\pi_i \geq \omega_i$. Thus, the binding participation constraint will not necessarily be that of the lowest ability agent, but of agent $k \in \arg\min_{1 \leq i \leq n}(\pi_i - \omega_i)$. Depending on who that agent is, the optimal allocation of prizes can be the same or quite different from what we describe above. Specifically, nothing will change if $a_k < 0$, but if $a_k > 0$, the optimal j-tournament will have a j that *maximizes*, as opposed to minimizes, the ratio Λ_r/B_r. Thus, if the configuration of outside options is such that the participation constraint is binding for one of the high-ability agents, optimal contracts may shift in the direction of winner-prize tournaments, i.e., those that award a high prize to relatively few top performers, because now it is the top performers whose incentives

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[19]Proposition 2 predicts that the optimal j is given by the $r \in \{1, 2, 3\}$ that minimizes Λ_r/B_r. In our example, $\Lambda_1/B_1 = 0.714$, $\Lambda_2/B_2 = 0$ and $\Lambda_3/B_3 = -0.714$.

are critical.

Kräkel (2012) assumes that the outside option is positively correlated with ability. We can explore the effect of such correlation by letting $\kappa_i = \kappa a_i$, where $\kappa > 0$ is some coefficient. Recall that agent i's payoff, in the linear approximation, is $\pi_i = \bar{\pi} + \eta(\bar{e})a_i$, where $\bar{\pi}$ is the payoff in the symmetric equilibrium. This gives $\pi_i - \omega_i = \bar{\pi} - \omega + (\eta(\bar{e}) - \kappa)a_i$. Thus, if κ is small compared to $\eta(\bar{e})$, there will be no effect on optimal contracts. If κ is large compared to η, optimal contracts will be reversed (i.e., focusing on winner prizes instead of loser prizes). The nontrivial case is when κ is close to η in magnitude. Recall that η can be manipulated through the structure of prizes; at the same time, the optimal prize structure will depend on the sign of $\eta - \kappa$.

We conclude that the presence of heterogeneity in outside options does not change the basic j-tournament structure of optimal contracts; moreover, it does not change the optimality of loser-prize contracts as long as the variation in outside options is small compared to variation in ability. Nontrivial reversals of optimal contracts in the direction of winner-prize contracts may occur, however, if the variation in outside options is relatively strong.

5 Discussion and conclusions

In this paper, we address the following question: If a firm uses relative performance evaluation-based incentives, for example, to decide on bonuses, promotion/demotion or salary raises, which prize structure is most effective? We use a standard principal-agent model of tournaments that yields the same levels of aggregate effort and firm's profit for various prize structures when workers are homogeneous in ability. For heterogeneous workers, however, the equivalence of multiple prize allocations is broken. We show that it is never optimal to just reward the best performer, and that under a wide range of conditions optimal contracts are those emphasizing loser prizes, i.e., awarding a low prize to the relatively few worst-performing employees. The result follows from the effect different prize allocations have on the degree of discouragement of low-ability workers.

We also show that the efficiency of anonymous tournament contracts (i.e., contracts in which prizes can only be conditioned on the ranking of output but not on the individual worker's ability) is robust to heterogeneity as long as heterogeneity is not too strong. The inefficiency of such contracts is a second-order effect in the level of heterogeneity, while the differences in firms' profits across different prize allocations are of the first-order.

Our results complement those of Akerlof and Holden (2012) and Moldovanu et al. (2012), who study optimal tournament contracts, respectively, for homogeneous agents

in the presence of risk aversion and in an all-pay auction setting under incomplete information about agents' abilities. We show that the agents' *ex ante* heterogeneity is an independent channel through which optimal prize schedules can be pinned down.

When workers within an organization observe their peers' compensation, they may form expectations and social reference points which have been shown empirically to have an impact on work effort and job satisfaction (see, e.g., Abeler et al. 2011, Card et al. 2012). From this perspective, winner-prize and loser-prize compensation profiles can be viewed, respectively, as tournaments with "reward" and "punishment".[20] Indeed, it can be argued that a contract awarding one or few low prizes is interpreted by workers as a contract with punishment because the high prize received by the majority of workers is perceived as an expectation or a norm, which implies that receiving the low prize is a signal that this worker was singled out as being the worst and hence "punished" even if she receives a positive prize. Think, for example, of a situation when all workers receive a 3% raise except for few who receives a 1% raise. Similarly, if only one or few best-performing workers receive a high prize they may feel singled out in a positive way, i.e., "rewarded," whereas the majority receiving a low prize are likely to think of their performance as standard or normal and their prize as expected. Thus, our results can also be interpreted as a comparison between tournament pay schemes emphasizing reward and punishment relative to a social reference point.

In this paper, we restricted attention to the case when all n agents participate in the tournament. An interesting issue that arises in tournaments of heterogeneous agents is that of worker exclusion. That is, under what circumstances is it optimal to implement a contract that does not satisfy the participation constraint of the lowest-ability agent(s)? By excluding these agents, the firm loses the revenue from their output but gains in terms of average output per worker and total compensation. However, as we show formally in Appendix A.4, it is never optimal to exclude workers when agents are weakly heterogeneous. The reason is that the gain from exclusion is $O(\mu)$ whereas the loss is $O(1)$, i.e., the latter is parametrically larger. This would not necessarily be the case, however, when agents are strongly heterogeneous, and an exploration of the resulting optimal tournament size is an interesting direction for future research.

Our analysis has several limitations dropping which can be of interest in terms of possible extensions. First, we assume that workers are risk-neutral. More complex incentive schemes involving more than two distinct prizes can be optimal under risk-aversion (Ak-

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[20]Nalebuff and Stiglitz (1983) used the term "penalties" to describe tournaments in which the worst-performing employee receives a low (but positive) prize and all others receive a high prize. This categorization is different from the one adopted by Moldovanu et al. (2012) who use the term "punishment" to denote a negative prize.

erlof and Holden 2012). Further steps in this direction include considering workers with heterogeneous risk attitudes and/or with preferences departing from the expected utility theory. Second, we restricted attention to the case of relatively weak heterogeneity. Although the impact of heterogeneity on aggregate effort is a second-order effect compared to the first-order effect of heterogeneity on the optimal aggregate compensation, it can become large and surpass the latter in magnitude when heterogeneity becomes strong. As discussed in the Introduction, this effect is likely to be mitigated by endogenous sorting of employees; nevertheless, it may be of interest to explore the interplay between possible nonlinear gains from strong heterogeneity in terms of aggregate effort and losses in terms of aggregate compensation. Third, we follow the tradition of Lazear and Rosen (1981) and effectively collapse the dynamic nature of employment into one decision-making period. A richer model can study explicitly the multi-period principal-agent interaction and the role of prize structures (including termination) in the optimal provision of incentives in a dynamic setting.

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A   Proofs of propositions

A.1   Proof of Proposition 1

We first prove the following two lemmas.

Lemma 1 *The expression for λ_r is given by Eq. (8).*

Proof. By definition, $\lambda_r = p_{11}^{(1,r)} - p_{12}^{(1,r)}$. Suppose agent 1 exerts effort e_1 and all agents $j \geq 2$ exert effort $\bar{e}$. The probability of player 1 being ranked r can be written as

$$p^{(1,r)}(e_1, \bar{e}, \ldots, \bar{e}) = \binom{n-1}{r-1} \int F(t + e_1 - \bar{e})^{n-r}[1 - F(t + e_1 - \bar{e})]^{r-1} f(t)dt. \quad (15)$$

Then $p_{11}^{(1,r)}$ can be found by differentiating Eq. (15) twice with respect to e_1 and then setting $e_1 = \bar{e}$. For convenience, we will use the following notation for the integrals arising in this calculation:

$$M_k = \int F(t)^{n-k}[1 - F(t)]^{k-3} f(t)^3 dt.$$

24

Equation (15) then gives

$$p_{11}^{(1,r)} = \binom{n-1}{r-1} \frac{\partial}{\partial e_1} \left[\int F(t+e_1-\bar{e})^{n-r-1}[1-F(t+e_1-\bar{e})]^{r-2} \right.$$

$$\left. \times [(n-r)(1-F(t+e_1-\bar{e})) - (r-1)F(t+e_1-\bar{e})]f(t+e_1-\bar{e})f(t)dt \right]_{e_1=\bar{e}}$$

$$= \binom{n-1}{r-1} \left[(n-r-1)[(n-r)M_{r+2} - (r-1)M_{r+1}] - (r-2)[(n-r)M_{r+1} - (r-1)M_r] \right.$$

$$\left. -(n-1)M_{r+1} + \int F(t)^{n-r-1}[1-F(t)]^{r-2}[(n-r)(1-F(t)) - (r-1)F(t)]f'(t)f(t)dt \right].$$

Suppose now that agents 1 and 2 exert efforts e_1 and e_2, respectively, and all agents $j \geq 3$ exert effort $\bar{e}$. The probability of player 1 being ranked r can be written as

$$p^{(1,r)}(e_1, e_2, \bar{e}, \ldots, \bar{e}) \tag{16}$$

$$= \binom{n-2}{r-1} \int F(t+e_1-\bar{e})^{n-r-1}F(t+e_1-e_2)[1-F(t+e_1-\bar{e})]^{r-1}f(t)dt$$

$$+ \binom{n-2}{r-2} \int F(t+e_1-\bar{e})^{n-r}[1-F(t+e_1-\bar{e})]^{r-2}[1-F(t+e_1-e_2)]f(t)dt.$$

Here, the first term is the probability that $y_1 > y_2$ and y_1 is ranked r among the remaining $n-1$ agents; and the second term is the probability that $y_1 < y_2$ and y_1 is ranked $r-1$ among the remaining $n-1$ agents. The expression for $p_{12}^{(1,r)}$ can be found by differentiating Eq. (16) with respect to e_1 and e_2 and then setting $e_1 = e_2 = \bar{e}$. This gives

$$p_{12}^{(1,r)} = \binom{n-2}{r-1} \frac{\partial}{\partial e_2} \left[(n-r-1) \int F(t)^{n-r-2}F(t+\bar{e}-e_2)[1-F(t)]^{r-1}f(t)^2 dt \right.$$

$$+ \int F(t)^{n-r-1}[1-F(t)]^{r-1}f(t+\bar{e}-e_2)f(t)dt$$

$$\left. -(r-1) \int F(t)^{n-r-1}F(t+\bar{e}-e_2)[1-F(t)]^{r-2}f(t)^2 dt \right]_{e_2=\bar{e}}$$

$$+ \binom{n-2}{r-2} \frac{\partial}{\partial e_2} \left[(n-r) \int F(t)^{n-r-1}[1-F(t)]^{r-2}[1-F(t+\bar{e}-e_2)]f(t)^2 dt \right.$$

$$-(r-2) \int F(t)^{n-r}[1-F(t)]^{r-3}[1-F(t+\bar{e}-e_2)]f(t)^2 dt$$

$$\left. - \int F(t)^{n-r}[1-F(t)]^{r-2}f(t+\bar{e}-e_2)f(t)dt \right]_{e_2=\bar{e}}$$

$$= \binom{n-2}{r-1}\left[-(n-r-1)M_{r+2} + (r-1)M_{r+1} - \int F(t)^{n-r-1}[1-F(t)]^{r-1}f'(t)f(t)dt\right]$$

$$+ \binom{n-2}{r-2}\left[(n-r)M_{r+1} - (r-2)M_r + \int F(t)^{n-r}[1-F(t)]^{r-2}f'(t)f(t)dt\right].$$

In addition to the various M_k terms, the expressions for $p_{11}^{(1,r)}$ and $p_{12}^{(1,r)}$ contain the integrals involving $f'(t)$. These integrals can be dealt with through integration by parts. Collecting the integrals from both expressions in $p_{11}^{(1,r)} - p_{12}^{(1,r)}$, obtain

$$\int \left[\binom{n-1}{r-1}F(t)^{n-r-1}[1-F(t)]^{r-2}[(n-r)(1-F(t)) - (r-1)F(t)]\right.$$

$$\left. + \binom{n-2}{r-1}F(t)^{n-r-1}[1-F(t)]^{r-1} - \binom{n-2}{r-2}F(t)^{n-r}[1-F(t)]^{r-2}\right]f'(t)f(t)dt$$

$$= n\int \left[\binom{n-2}{r-1}F(t)^{n-r-1}[1-F(t)]^{r-1} - \binom{n-2}{r-2}F(t)^{n-r}[1-F(t)]^{r-2}\right]f'(t)f(t)dt$$

$$= \Delta_r - \frac{n}{2}\binom{n-2}{r-1}[(n-r-1)M_{r+2} - (r-1)M_{r+1}] + \frac{n}{2}\binom{n-2}{r-2}[(n-r)M_{r+1} - (r-2)M_r].$$

Here, Δ_r is the part determined by the boundary values of the distribution of noise:

$$\Delta_r = \frac{n}{2}\binom{n-2}{r-1}F(t)^{n-r-1}[1-F(t)]^{r-1}f(t)^2\Big|_{u_l}^{u_h} - \frac{n}{2}\binom{n-2}{r-2}F(t)^{n-r}[1-F(t)]^{r-2}f(t)^2\Big|_{u_l}^{u_h}.$$

It is easy to see that Δ_r is equal to zero except for some special values of n and r. Specifically, for $n = 2$, we have

$$\Delta_1 = -\Delta_2 = f(u_h)^2 - f(u_l)^2,$$

while for $n \geq 3$,

$$\Delta_1 = \frac{nf(u_h)^2}{2}, \quad \Delta_2 = -\frac{n}{2}(f(u_l)^2 I_{n=3} + f(u_h)^2),$$

$$\Delta_{n-1} = -\frac{n}{2}(f(u_l)^2 + f(u_h)^2 I_{n=3}), \quad \Delta_n = \frac{nf(u_l)^2}{2}.$$

In all other cases, $\Delta_r = 0$.

Going back to the expression for $p_{11}^{(1,r)} - p_{12}^{(1,r)}$, we can now collect all the remaining terms:

$$p_{11}^{(1,r)} - p_{12}^{(1,r)} = \Delta_r + \binom{n-1}{r-1}(n-r-1)(n-r)M_{r+2} - \binom{n-1}{r-1}(n-r-1)(r-1)M_{r+1}$$

$$-\binom{n-1}{r-1}(n-r)(r-2)M_{r+1}+\binom{n-1}{r-1}(r-1)(r-2)M_r-\binom{n-1}{r-1}(n-1)M_{r+1}$$

$$-\frac{n}{2}\binom{n-2}{r-1}(n-r-1)M_{r+2}+\frac{n}{2}\binom{n-2}{r-1}(r-1)M_{r+1}+\binom{n-2}{r-1}(n-r-1)M_{r+2}$$

$$-\binom{n-2}{r-1}(r-1)M_{r+1}-\binom{n-2}{r-2}(n-r)M_{r+1}+\binom{n-2}{r-1}(r-2)M_r$$

$$+\frac{n}{2}\binom{n-2}{r-2}(n-r)M_{r+1}-\frac{n}{2}\binom{n-2}{r-2}(r-2)M_r.$$

Collecting the terms with M_r, M_{r+1} and M_{r+2} and using the properties of binomial coefficients, finally obtain Eq. (8).

Q.E.D.

Lemma 2

$$p_2^{(1,r)}=-\frac{\beta_r}{n-1}.$$

Proof. The expression for $p_2^{(1,r)}$ can be obtained by differentiating Eq. (16) with respect to e_2 and setting $e_1=e_2=\bar{e}$. This gives

$$p_2^{(1,r)}=-\binom{n-2}{r-1}\int F(t)^{n-r-1}[1-F(t)]^{r-1}f(t)^2dt+\binom{n-2}{r-2}\int F(t)^{n-r}[1-F(t)]^{r-2}f(t)^2dt$$

$$=\frac{(n-2)!}{(r-2)!(n-r-1)!}\int F(t)^{n-r-1}[1-F(t)]^{r-2}\left(-\frac{1-F(t)}{r-1}+\frac{F(t)}{n-r}\right)f(t)^2dt$$

$$=\frac{(n-2)!}{(r-1)!(n-r)!}\int F(t)^{n-r-1}[1-F(t)]^{r-2}[-(n-r)(1-F(t))+(r-1)F(t)]f(t)^2dt$$

$$=-\frac{\beta_r}{n-1}.$$

Q.E.D.

We now go back to the proof of Proposition 1. For part (a), start by plugging the representations $e_i=\bar{e}(1+x_i)$ and $c_i=\bar{c}(1-a_i)$ into Eq. (5):

$$\sum_r p_i^{(i,r)}(\bar{e}(1+x_1),\ldots,\bar{e}(1+x_n))V_r=\bar{c}(1-a_i)g'(\bar{e}(1+x_i)).$$

The next step is to expand both sides of the equation in Taylor series to the first order in μ treating x_i and a_i as small corrections linear in μ. The left-hand side becomes

$$\sum_r(p_i^{(i,r)}+p_{ii}^{(i,r)}\bar{e}x_i+\sum_{j\neq i}p_{ij}^{(i,r)}\bar{e}x_j)V_r+O(\mu^2).$$

Here and below, all the derivatives of $p^{(i,r)}$ are evaluated at the symmetric equilibrium point $(\bar{e}, \ldots, \bar{e})$. Note that, by symmetry, $p_i^{(i,r)} = p_1^{(1,r)} \equiv \beta_r$ for all i and, likewise, $p_{ii}^{(i,r)} = p_{11}^{(1,r)}$ for all i and $p_{ij}^{(i,r)} = p_{12}^{(1,r)}$ for all $i \neq j$. Introducing $X = \sum_i x_i$, finally obtain for the left-hand side of (5),

$$\sum_r (\beta_r + \lambda_r \bar{e} x_i + p_{12}^{(1,r)} \bar{e} X) V_r + O(\mu^2).$$

Here, $\lambda_r \equiv p_{11}^{(1,r)} - p_{12}^{(1,r)}$.

Similarly expanding the right-hand side of (5), obtain

$$c_i g'(e_i) = \bar{c}(1 - a_i) g'(\bar{e}(1 + x_i)) = \bar{c}[g'(\bar{e}) + g''(\bar{e})\bar{e} x_i - g'(\bar{e})a_i] + O(\mu^2).$$

Equating the two expressions and using Eq. (1), obtain

$$\sum_r (\lambda_r \bar{e} x_i + p_{12}^{(1,r)} \bar{e} X) V_r = \bar{c}\bar{e}g''(\bar{e}) x_i - \bar{c}g'(\bar{e})a_i + O(\mu^2).$$

Summing this expression over i and using the restriction $\sum_i a_i = 0$ gives $X = O(\mu^2)$, which, together with Eq. (1), produces Eq. (6).

For part (b), write agent i's equilibrium payoff, $\pi_i = \sum_r p^{(i,r)} V_r - c_i g(e_i)$, using the representations $c_i = \bar{c}(1 - a_i)$ and $e_i = \bar{e}(1 + x_i)$:

$$\pi_i = \sum_r p^{(i,r)}(\bar{e}(1 + x_1), \ldots, \bar{e}(1 + x_n)) V_r - \bar{c}(1 - a_i) g(\bar{e}(1 + x_i)).$$

Expanding this expression to the first order in μ, obtain

$$\pi_i = \sum_r \left(\frac{1}{n} + p_i^{(i,r)} \bar{e} x_i + \sum_{j \neq i} p_j^{(i,r)} \bar{e} x_j \right) V_r - \bar{c}(g(\bar{e}) + g'(\bar{e})\bar{e} x_i - a_i g(\bar{e})) + O(\mu^2)$$

$$= \frac{1}{n} \sum_r V_r - \bar{c}g(\bar{e}) + \bar{c}g(\bar{e})a_i - \bar{e} \sum_r p_2^{(1,r)} V_r x_i + O(\mu^2).$$

Using Lemma 2, this can be written as

$$\pi_i = \frac{1}{n} \sum_r V_r - \bar{c}g(\bar{e}) + \bar{c}g(\bar{e})a_i + \frac{\bar{e} \sum_r \beta_r V_r x_i}{n - 1} + O(\mu^2).$$

which, together with Eqs. (1) and (6), gives the result.

A.2   Proof of Proposition 2

The principal's profit is $\Pi = n[\bar{e} - \bar{c}g(\bar{e}) - \omega + \eta(\bar{e})a_n] + O(\mu^2)$. Let $\bar{e} = \bar{e}^s + \tau a_n$. In the linear approximation, the optimal profit Π can be evaluated at $\bar{e} = \bar{e} = \bar{e}^s$, due to the envelope theorem; hence part (c) of the proposition.

For part (a), note that $\eta(\bar{e}^s)$ is evaluated at the parameters of the symmetric optimal contract, which gives (cf. Eq. (7) and the fact that $\bar{c}g'(\bar{e}^s) = 1$)

$$\eta(\bar{e}^s) = \frac{1}{(n-1)[\bar{c}g''(\bar{e}) - \sum_r \lambda_r \bar{V}_r]} + \bar{c}g(\bar{e}^s).$$

By construction the relative ability of agent n is negative, $a_n < 0$, therefore the principal will choose the prize structure $(\bar{V}_1, \ldots, \bar{V}_n)$ that minimizes the loss term in the profit, $n\eta(\bar{e}^s)|a_n|$, i.e., minimizes $\eta(\bar{e}^s)$. This leads to the following principal's problem:

$$\min_{\bar{V}_1, \ldots, \bar{V}_n} \sum_r \lambda_r \bar{V}_r \quad \text{s.t.} \quad \sum_r \beta_r \bar{V}_r = 1, \quad \sum_r \bar{V}_r = n[\omega + \bar{c}g(\bar{e}^s)].$$

Let $D_r = \bar{V}_r - \bar{V}_{r+1}$ for $r = 1, \ldots, n-1$ denote the differences between adjacent prizes. By construction, $D_r \geq 0$. Prizes $\bar{V}_r$ can then be written as $\bar{V}_r = \sum_{j=r}^{n-1} D_j + \bar{V}_n$. This gives

$$\sum_r \beta_r \bar{V}_r = \beta_1(D_1 + \ldots + D_{n-1} + \bar{V}_n) + \beta_2(D_2 + \ldots + D_{n-1} + \bar{V}_n) + \ldots + \beta_n \bar{V}_n$$

$$= \beta_1 D_1 + (\beta_1 + \beta_2)D_2 + \ldots + (\beta_1 + \ldots + \beta_n)\bar{V}_n = \sum_{r=1}^{n-1} B_r D_r,$$

where $B_r = \sum_{j=1}^r \beta_r$; we also used the fact that $B_n = 0$. Similarly,

$$\sum_r \lambda_r \bar{V}_r = \sum_{r=1}^{n-1} \Lambda_r D_r, \quad \sum_r \bar{V}_r = \sum_{r=1}^{n-1} r D_r + n\bar{V}_n.$$

Here $\Lambda_r = \sum_{j=1}^r \lambda_r$, with $\Lambda_n = 0$.

The principal's problem can be written in terms of the variables D_r as

$$\min_{D_1, \ldots, D_{n-1} \geq 0} \sum_{r=1}^{n-1} \Lambda_r D_r \quad \text{s.t.} \quad \sum_{r=1}^{n-1} B_r D_r = 1. \tag{17}$$

Note that the second constraint is no longer relevant for the minimization problem and only serves to determine the lowest prize: $\bar{V}_n = \omega + \bar{c}g(\bar{e}^s) - (1/n)\sum_{r=1}^{n-1} r D_r$.

The following lemma shows that the cumulative coefficients B_r and Λ_r are indeed

given by Eqs. (11).

Lemma 3 $B_r = \sum_{k=1}^r \beta_k$ and $\Lambda_r = \sum_{k=1}^r \lambda_k$ are given by Eqs. (11).

Proof. It easy to see that for $r = 1$ both formulas are correct. It is, therefore, sufficient to show that $\beta_r = B_r - B_{r-1}$ and $\lambda_r = \Lambda_r - \Lambda_{r-1}$, with B_r and Λ_r given by Eqs. (11). We have

$$B_r - B_{r-1} = \frac{(n-1)!}{(n-r-1)!(r-1)!} \int F(t)^{n-r-1}[1-F(t)]^{r-1}f(t)^2 dt$$

$$- \frac{(n-1)!}{(n-r)!(r-2)!} \int F(t)^{n-r}[1-F(t)]^{r-2}f(t)^2 dt$$

$$= \frac{(n-1)!}{(n-r)!(r-1)!} \int F(t)^{n-r-1}[1-F(t)]^{r-2}[(n-r)(1-F(t))-(r-1)F(t)]f(t)^2 dt = \beta_r;$$

$$\Lambda_r - \Lambda_{r-1} = \frac{n(n-2)!}{2(n-r-1)!(r-1)!}[(n-r-1)M_{r+2}-(r-1)M_{r+1}]$$

$$- \frac{n(n-2)!}{2(n-r)!(r-2)!}[(n-r)M_{r+1}-(r-2)M_r] + \Delta_r$$

$$= \frac{n(n-2)!}{2(n-r)!(r-1)!}[(n-r)(n-r-1)M_{r+2}-2(n-r)(r-1)M_{r+1}+(r-1)(r-2)M_r]+\Delta_r = \lambda_r.$$

Q.E.D.

It follows from (11) that coefficients B_r are positive for all $r < n$. Thus, the constraints of problem (17) define a convex polygon whose vertices $k = 1, \ldots, n-1$ have $D_k = 1/B_k$ and $D_r = 0$ for all $r \neq k$. The objective function is linear, therefore the minimum will be reached at one of the vertices.[21] Specifically, an optimal vertex is $j \in \arg\min_{1 \leq r \leq n-1} \Lambda_r/B_r$.

Thus, the optimal prize structure is such that $D_j = 1/B_j$ for some j and $D_r = 0$ for $r \neq j$. The nth prize, therefore, is $\bar{V}_n = \omega + \bar{c}g(\bar{e}^s) - j/nB_j$. This leads to the following optimal configuration of symmetric prizes: $\bar{V}_1 = \ldots = \bar{V}_j = \bar{V}_n + 1/B_j$ and $\bar{V}_{j+1} = \ldots = \bar{V}_n$.

Now that the optimal structure of symmetric prizes is determined, we are in a position to find the optimal prizes W_1 and W_2, assuming the principal implements average equilibrium effort $\bar{e}^s$,[22] for weakly heterogeneous agents (part (b)).

The optimal prizes W_1 and W_2 satisfy the equations $\sum_r \beta_r V_r = \bar{c}g'(\bar{e})$ and $\pi_n = \omega$.

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[21]It is possible to have multiple minima when $\Lambda_k/B_k = \Lambda_l/B_l$ for some $k \neq l$. Such solutions are nongeneric; besides, any of the optimal vertices can be used as a solution anyway.

[22]For a more general case, see the next section.

With $\bar{e} = \bar{e}^s$ and the j-tournament prize structure, these become,

$$B_j(W_1 - W_2) = 1,$$

$$\frac{jW_1}{n} + \frac{(n-j)W_2}{n} - \bar{c}g(\bar{e}^s) + \eta(\bar{e}^s)a_n = \omega.$$

Solving this system of equations, obtain the result.

A.3 Implementing average equilibrium effort different from $\bar{e}^s$

Suppose the principal would like to implement average equilibrium effort $\bar{e} = \bar{e}^s + \epsilon$, where $\epsilon = O(\mu)$. With appropriately adjusted prizes, this will have no effect on the principal's profit or total surplus, to the first order in μ.

The optimal prizes W_1 and W_2 must satisfy the equations $\sum_r \beta_r V_r = \bar{c}g'(\bar{e})$ and $\pi_n = \omega$. With the j-tournament prize structure and $\bar{e} = \bar{e}^s + \epsilon$, these become, in the linear approximation,

$$B_j(W_1 - W_2) = 1 + \bar{c}g''(\bar{e}^s)\epsilon,$$

$$\frac{jW_1}{n} + \frac{(n-j)W_2}{n} - \bar{c}g(\bar{e}^s) - \epsilon + \eta(\bar{e}^s)a_n = \omega.$$

Solving this system of equations, obtain,

$$W_1 = \omega + \bar{c}g(\bar{e}^s) + \frac{n-j}{nB_j} - \eta(\bar{e}^s)a_n + \left[1 + \frac{(n-j)\bar{c}g''(\bar{e}^s)}{nB_j}\right]\epsilon + O(\mu^2),$$

$$W_2 = \omega + \bar{c}g(\bar{e}^s) - \frac{j}{nB_j} - \eta(\bar{e}^s)a_n + \left[1 - \frac{j\bar{c}g''(\bar{e}^s)}{nB_j}\right]\epsilon + O(\mu^2).$$

Note that such a contract can, in principle, be introduced even for symmetric agents, i.e., for $a_n = 0$, and it will be linearly efficient. Of course, in the case of symmetric agents we know that the second-order effect of a deviation of implemented effort from $\bar{e}^s$ is unambiguously negative; but it is interesting that this kind of arbitrariness is permissible if we only care about linear effects. In the case of heterogeneous agents, the direction of the second-order effect is not as obvious, but its analysis goes beyond the scope of this paper.

A.4 Excluding an agent

As discussed in Section 5, it may be of interest to explore whether the principal will choose a contract that satisfies the participation constraints of all n agents or she will sometimes prefer to exclude the lowest-ability agent(s). There are two competing effects of such an

exclusion: A loss of output (and revenue) from the excluded agent(s) and a gain from the lower total payment to the remaining agents. As we show in this section, while the loss is of the zeroth order in μ, the gain is of the first order in μ, i.e., the gain is much smaller than the loss. Moreover, the gain does not increase with n. Thus, when agents are weakly heterogeneous it is never optimal for the principal to exclude agents.

Consider a tournament of n agents with cost parameters $(c_1, \ldots, c_n)$ ordered so that $c_1 \leq \ldots \leq c_n$, i.e., agent n is the lowest-ability agent. Let $\bar{c}_n = \frac{1}{n} \sum_{i=1}^{n} c_i$ denote the average cost parameter among the n agents. The average equilibrium effort, $\bar{e}_n^s$, satisfies the equation $\bar{c}_n g'(\bar{e}_n^s) = 1$. The firm's profit, cf. Proposition 2(c), is, in the linear approximation,

$$\Pi_n = n[\bar{e}_n^s - \bar{c}_n g(\bar{e}_n^s) - \omega] + n\eta(\bar{e}_n^s)a_n^{(n)},$$

where $a_n^{(n)}$ is the relative ability of agent n in the tournament of n agents, defined by $c_n = \bar{c}_n(1 - a_n^{(n)})$.

Consider now the tournament of $n - 1$ agents where agent n has been excluded. The average cost parameter is now $\bar{c}_{n-1} = \frac{1}{n-1} \sum_{i=1}^{n-1} c_i$, and can be written as

$$\bar{c}_{n-1} = \frac{1}{n-1} \sum_{i=1}^{n-1} c_i = \frac{n\bar{c}_n - c_n}{n-1} = \frac{n\bar{c}_n - \bar{c}_n(1 - a_n^{(n)})}{n-1} = \bar{c}_n \left(1 + \frac{a_n^{(n)}}{n-1}\right).$$

As expected, $\bar{c}_{n-1}$ is lower than $\bar{c}_n$, and the difference is $O(\mu)$. Importantly, the difference decreases in n. The new average equilibrium effort, $\bar{e}_{n-1}^s$, satisfies the equation $\bar{c}_{n-1} g'(\bar{e}_{n-1}^s) = 1$. Writing $\bar{e}_{n-1}^s = \bar{e}_n^s + \delta$, we have

$$\bar{c}_n \left(1 + \frac{a_n^{(n)}}{n-1}\right) g'(\bar{e}_n^s + \delta) = 1,$$

therefore, in the linear approximation,

$$\delta = -\frac{a_n^{(n)}}{(n-1)\bar{c}_n g''(\bar{e}_n^s)}.$$

The profit of the firm with $n - 1$ agents is

$$\Pi_{n-1} = (n-1)[\bar{e}_{n-1}^s - \bar{c}_{n-1} g(\bar{e}_{n-1}^s) - \omega] + (n-1)\eta(\bar{e}_{n-1}^s)a_{n-1}^{(n-1)},$$

where $a_{n-1}^{(n-1)}$ is the relative ability of agent $n-1$ in the tournament of $n-1$ agents, defined

32

by $c_{n-1} = \bar{c}_{n-1}(1 - a_{n-1}^{(n-1)})$, which can be rewritten as

$$a_{n-1}^{(n-1)} = 1 - \frac{c_{n-1}}{\bar{c}_{n-1}} = 1 - \frac{\bar{c}_n(1 - a_{n-1}^{(n)})}{\bar{c}_{n-1}} = 1 - \frac{1 - a_{n-1}^{(n)}}{1 + \frac{a_n^{(n)}}{n-1}}.$$

Here, $a_{n-1}^{(n)}$ is the relative ability of agent $n-1$ in the tournament of n agents. In the linear approximation, the equation above gives

$$a_{n-1}^{(n-1)} = a_{n-1}^{(n)} + \frac{a_n^{(n)}}{n-1}.$$

The difference in profits between the tournaments of n and $n-1$ agents is, therefore,

$$\Pi_n - \Pi_{n-1} = n[\bar{e}_n^s - \bar{c}_n g(\bar{e}_n^s) - \omega] + n\eta(\bar{e}_n^s)a_n^{(n)}$$

$$-(n-1)\left[\bar{e}_n^s + \delta - \bar{c}_n\left(1 + \frac{a_n^{(n)}}{n-1}\right)g(\bar{e}_n^s + \delta) - \omega\right] - (n-1)\eta(\bar{e}_n^s + \delta)\left(a_{n-1}^{(n)} + \frac{a_n^{(n)}}{n-1}\right).$$

In the linear approximation, we have

$$\bar{c}_n\left(1 + \frac{a_n^{(n)}}{n-1}\right)g(\bar{e}_n^s + \delta) = \bar{c}_n g(\bar{e}_n^s) + \frac{\bar{c}_n g(\bar{e}_n^s)a_n^{(n)}}{n-1} + \delta,$$

$$\eta(\bar{e}_n^s + \delta)\left(a_{n-1}^{(n)} + \frac{a_n^{(n)}}{n-1}\right) = \eta(\bar{e}_n^s)\left(a_{n-1}^{(n)} + \frac{a_n^{(n)}}{n-1}\right).$$

This gives

$$\Pi_n - \Pi_{n-1} = n[\bar{e}_n^s - \bar{c}_n g(\bar{e}_n^s) - \omega] + n\eta(\bar{e}_n^s)a_n^{(n)}$$

$$-(n-1)\left[\bar{e}_n^s - \bar{c}_n g(\bar{e}_n^s) - \frac{\bar{c}_n g(\bar{e}_n^s)a_n^{(n)}}{n-1} - \omega\right] - (n-1)\eta(\bar{e}_n^s)a_{n-1}^{(n)} - \eta(\bar{e}_n^s)a_n^{(n)}$$

$$= [\bar{e}_n^s - \bar{c}_n g(\bar{e}_n^s) - \omega] + (n-1)\eta(\bar{e}_n^s)(a_n^{(n)} - a_{n-1}^{(n)}) + \bar{c}_n g(\bar{e}_n^s)a_n^{(n)}.$$

The first term (in the square brackets) in the expression above is positive and large $(O(1))$. The second term is $O(\mu)$ and it does not grow with n because $a_n^{(n)} - a_{n-1}^{(n)}$ is $\sim \frac{O(\mu)}{n}$. Finally, the third term is $O(\mu)$. The first term represents the loss, while the second and third terms (by absolute value) represent the gain from excluding the lowest-ability agent. Thus, in the case of weakly heterogeneous agents the loss is parametrically larger than the gain and hence $\Pi_n - \Pi_{n-1}$ is always positive, i.e., it is not optimal to exclude agents.

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Loukas Balafoutas, E. Glenn Dutcher, Florian Lindner, Dmitry Ryvkin

The optimal allocation of prizes in tournaments of heterogeneous agents

Abstract
Tournaments are widely used in organizations, explicitly or implicitly, to reward the best-performing employees, e.g., through promotion or bonuses, and to punish the worst-performing employees, e.g., through firing or unfavorable job assignments. We use a principal-agent model to compare the efficiency of two tournament incentive schemes, reward tournament and punishment tournament, which, respectively, reward the best performer and punish the worst performer. We show that while the two schemes are equivalent when agents are symmetric in their ability, the equivalence is broken in the presence of heterogeneity. Specifically, punishment tournaments lead to higher profits of the firm. The reason is that low-ability agents are discouraged less in punishment tournaments than in reward tournaments, and hence can be compensated less to meet their participation constraints. Hence, our results predict that firms using punishment tournament contracts will perform better.