prefmod: news and extensions

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Part I: Introduction

- R-Package **prefmod**
  collection of utilities to fit a variety of paired comparison models
- **preference models** based on paired comparisons
  objective is to establish a preference scale for certain objects
  – food, crimes, pain, teaching styles, portfolios, . . .
- **paired comparisons**
  J objects are compared in pairs according to a specific attribute
  – tastes better, makes me put on more weight, . . .

  we observe $\binom{J}{2}$ comparisons (responses)

Model

core model in prefmod is the Bradley-Terry specification

$$P\{Y_{jk}=1|\pi_j,\pi_k\}=\frac{\pi_j}{\pi_j+\pi_k} \quad \text{or} \quad P\{Y_{jk}=-1|\pi_j,\pi_k\}=\frac{\pi_k}{\pi_j+\pi_k}$$

$Y_{jk}=1$ . . . object j preferred to k, $Y_{jk}=-1$ . . . object k preferred to j
π_j . . . location of object j on preference scale

independence model (Bradley-Terry): response is y_{jk}

$$p(y_{jk})=c\left(\frac{\sqrt{\pi_j}}{\sqrt{\pi_k}}\right)^{y_{jk}}$$

pattern model: response is $y=\{y_{12},y_{13},\ldots,y_{jk},\ldots,y_{J-1,J}\}$

$$p(y_{12},\ldots,y_{J-1,J})=c\prod_{j<k}\left(\frac{\sqrt{\pi_j}}{\sqrt{\pi_k}}\right)^{y_{jk}}$$

Independence: LLBT (loglinear Bradley-Terry model)

we use the loglinear representation *(Applied Statistics, 1998)*

$$\ln m_{(y_{jk})}=\mu_{(jk)}+y_{jk}(\lambda_j-\lambda_k)$$

design structure for 3 objects:

			μ	λ_1	λ_2	λ_3
comparison	*decision*	counts	const	y_{12}	y_{13}	y_{23}
(12)	O_1	$n_{(1>2)}$	1	1	-1	0
(12)	O_2	$n_{(2>1)}$	1	-1	1	0
(13)	O_1	$n_{(1>3)}$	2	1	0	-1
(13)	O_3	$n_{(3>1)}$	2	-1	0	1
(23)	O_2	$n_{(2>3)}$	3	0	1	-1
(23)	O_3	$n_{(3>2)}$	3	0	-1	1

factor for normalizing constants μ

Pattern model

loglinear model *(CSDA, 2002)*

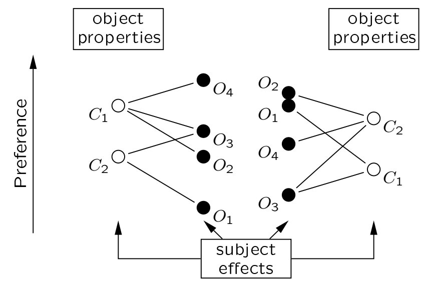
$$\ln m(y_{12},\ldots,y_{J-1,J}) = \eta_y = \mu + \sum_{j=1}^{J} \lambda_j x_j = \mu + \sum_{j=1}^{J} \lambda_j \left(\sum_{\nu=j+1}^{J} y_{j\nu} - \sum_{\nu=1}^{j-1} y_{\nu j} \right)$$

design structure for 3 objects:

pattern	y_{12}	y_{13}	y_{23}	counts	μ const	λ_1 x_1	λ_2 x_2	λ_3 x_3
ℓ_1	1	1	1	n_1	1	2	0	-2
ℓ_2	1	1	−1	n_2	1	2	-2	0
ℓ_3	1	−1	1	n_3	1	0	0	0
ℓ_4	1	−1	−1	n_4	1	0	-2	2
ℓ_5	−1	1	1	n_5	1	0	2	-2
ℓ_6	−1	1	−1	n_6	1	0	0	0
ℓ_7	−1	−1	1	n_7	1	-2	2	0
ℓ_8	−1	−1	−1	n_8	1	-2	0	2

$x_j = \#(O_j$ is preferred in $\ell)$ - $\#(O_j$ not preferred in $\ell)$

Extensions for subject and object effects



subject effects: duplicate table for each covariate group s

object effects: $\lambda_j = \sum_q \beta_q^C x_{jq}$

b_{jq} . . . covariate for characteristic C_q

β_q^C . . . effect of characteristic C_q

Extensions: Overview

extensions for LLBT and pattern model

- undecided ($3^{\binom{J}{2}}$ different patterns), position effects
- subject covariates, object specific covariates

additional extensions for pattern models

we can give up the assumption of independent decisions

- dependence parameters $\theta_{(jk)(jl)}$ (interactions) for pairs of comparisons with one object in common

and we can also deal with various other response formats

- ranking data
- rating (Likert) data ("rankings with ties")
- piling, multiple responses, . . .

Derived paired comparisons:

Example: ranking with 3 objects

we transform rankings to paired comparisons

Data				comparison		
R	G	B	Response	RG	RB	GB
1	2	3	R>G>B	1	1	1
1	3	2	R>B>G	1	1	-1
-	-	-	-	1	-1	1
2	3	1	B>R>G	1	-1	-1
2	1	3	G>R>B	-1	1	1
-	-	-	-	-1	1	-1
3	1	2	G>B>R	-1	-1	1
3	2	1	B>G>R	-1	-1	-1

- number of possible patterns is $3! = 6$ compared to $2^{\binom{3}{2}} = 8$
- pattern model based on reduced number of different patterns
- using the LLBT leads to biased estimates for the λ's →

The LLBT in prefmod

- user-friendly function (restricted functionality):

```
llbtPC.fit(obj, nitems, formel = ~1, elim = ~1, resptype = "paircomp",
    obj.names = NULL, undec = FALSE)
```

- for more specialised models: generate a design matrix
  use gnm() or glm() to fit the model

```
llbt.design(data, nitems = NULL, objnames = "", objcovs = NULL,
    cat.scovs = NULL, num.scovs = NULL, casewise = FALSE, ...)
```

- calculate the π's (λ's) from the estimated model

```
llbt.worth(fitobj, outmat = "worth")
```

- plot the π's (λ's) from the llbt.worth() output

```
plotworth(worthmat, main = "Preferences", ylab = "Estimate",
    psymb = NULL, pcol = NULL, ylim = range(worthmat))
```


LLBT example: CEMS exchange program

students of the WU can study abroad visiting one of currently 17 CEMS universities

aim of the study:

- preference orderings of students for different locations
- identify reasons for these preferences

data:

- paired comparison responses for 6 selected CEMS (London, Paris, Milan, Barcelona, St.Gall, Stockholm)
- several subject covariates (e.g., gender, working status, language abilities, etc.)
- several object covariates (e.g., specialisation, region, etc.)

LLBT example: CEMS exchange program

- generate object covariates (dummy coding):

```
> LAT <- c(0, 1, 1, 0, 1, 0)
> EC <- c(1, 0, 1, 0, 0, 0)
> MS <- c(0, 1, 0, 0, 1, 0)
> FS <- c(0, 0, 0, 1, 0, 1)
```

- make a data frame for object covariates, name objects

```
> OBJ <- data.frame(LAT, EC, MS, FS)
> cities <- c("LO", "PA", "MI", "SG", "BA", "ST")
```

- make a design matrix

```
> des.n1 <- llbt.design(cpc, 6, objcovs = OBJ, cat.scovs = "SEX",
+     objnames = cities)
```


Example (cont'd)

- fit model using gnm()

```
> mod <- gnm(y ~ LAT + MS + FS + SEX:(LAT + MS + FS), eliminate = mu:SEX,
+     family = poisson, data = des.n1)
```

- model results

```
> mod
Call:

gnm(formula = y ~ LAT + MS + FS + SEX:(LAT + MS + FS), eliminate = mu:SEX,
    family = poisson, data = des.n1)

Coefficients of interest:
     LAT        MS        FS   LAT:SEX2   MS:SEX2   FS:SEX2
-0.74972   0.02355  -1.00742  -0.29634   0.27508   0.16457

Deviance:            1322.009
Pearson chi-squared: 1203.450
Residual df:         54
```

Example (cont'd)

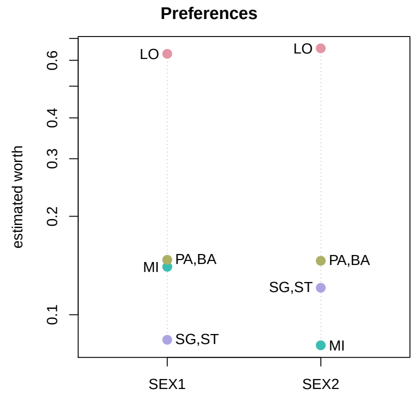
- calculate the worth

```
> wmat <- llbt.worth(mod)
> wmat
            SEX1       SEX2
LO    0.62868639 0.65230770
PA,BA 0.14712617 0.14629880
MI    0.14035778 0.08051178
SG,ST 0.08382965 0.12088172
attr(,"objtable")
  LAT MS FS      x
1   0  0  0     LO
2   1  0  0     MI
3   1  1  0 PA, BA
4   0  0  1 SG, ST
```

- plot the worth

```
> plotworth(wmat, ylab = "estimated worth", log = "y")
```


Example (cont'd)



The pattern model in prefmod

- user-friendly function (restricted functionality):
```
pattPC.fit(obj, nitems, formel = ~1, elim = ~1, obj.names = NULL,
    undec = FALSE, ia = FALSE)
```
- analogous for rankings (`pattR.fit`) and ratings (`pattL.fit`)
- calculate the π's (λ's) from the estimated model
```
patt.worth(obj, obj.names = NULL, outmat = "worth")
```
- plot the π's (λ's) from the `patt.worth()` output
```
plotworth(worthmat, main = "Preferences", ylab = "Estimate",
    psymb = NULL, pcol = NULL, ylim = range(worthmat))
```
- for more specialised models: generate a design matrix
```
patt.design(obj, nitems = NULL, objnames = "", resptype = "paircomp",
    blnRevert = FALSE, cov.sel = "", blnIntcovs = FALSE)
```


Part II: Model Extensions

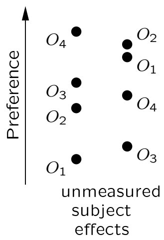
- heterogeneity in paired comparisons (latent classes)
  (Annals of Applied Statistics, 2010)
- missing observations
  (under revision, 2011)
- multivariate responses in the LLBT:
  multidimensional paired comparisons
  repeated measurements
  (being written)

Extension 1: Heterogeneity in paired comparisons

- responses vary between respondents
- measured covariates can be taken into account
- other unmeasured or unmeasurable characteristics of the respondents might affect the response

in practice mainly 2 situations:

- unknown or not available subject variables
- very complex situations make model fit untractable



Random effects model

introduce random effects for each respondent (pattern ℓ)
we need J random effect components $\delta_{j\ell s}$
the linear predictor is

$$\eta_{\ell s} = \sum_{j<j} y_{jk;\ell s}(\lambda_{js} + \delta_{j\ell s} - \lambda_{ks} - \delta_{k\ell s})$$

location of preference parameter for item j will be shifted up or down for each response pattern in each subject covariate group

the likelihood becomes

$$L = \prod_{\ell s}\Big(\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} P(y_{\ell s}|\delta_{\ell s})\; g(\delta_{\ell s})\; d\delta_{1\ell s}\, d\delta_{2\ell s}\, \dots\, d\delta_{J-1;\ell s}\Big)^{n_{\ell s}}$$

where $g(\delta_{\ell s})$ is the multivariate probability density function or mixing distribution of the random effects vector.

Nonparametric approach

alternative approach (NPML, Aitkin, 1996):
replace multivariate distribution by series of mass point components with unknown probability and unknown location →

mass point approach is a mixture model, where multinomial (fixed effects) model is replaced by mixture of multinomials

if number of components is known, say R, we get R vectors of mass-points locations

$$\delta_r = (\delta_{1r}, \delta_{2r}, \dots, \delta_{J-1;r})$$

and unknown component probability q_r

The likelihood now becomes

$$L = \prod_{\ell s}\Big(\sum_{r=1}^{R} q_r\, P_{\ell sr}(\mathbf{y}_{\ell s}|\boldsymbol{\delta}_{\mathbf{r}})\Big)^{n_{\ell s}} \quad \text{where } \sum_{\ell} P_{\ell sr} = 1, \quad \forall\, s, r$$

Estimation

using the EM algorithm

view problem as missing data problem:
latent class membership indicator $z_{\ell sr} \in \{0, 1\}$ for each ℓs combination

$$z_{\ell sr} = 1 \quad \text{if} \quad \ell s \in r \qquad E(z_{\ell sr}) = w_{\ell sr}$$

$w_{\ell sr}$ are the posterior probabilities of class membership
$z_{\ell sr}$ is missing

- E-step:
recalculates the w's given current parameter estimates for the q's and λ's

- M-step:
maximises the multinomial likelihood w.r.t. λ's and δ's
carried out through loglinear model with weights $w_{\ell sr}$

The NPML model in prefmod

```
pattnpml.fit(
        formula,              # formula for fixed effects
        random = ~1,          # formula for random effects
        k = 1,                # number of mass-points (classes)
        design,               # design matrix
        tol = 0.5,            # to control the EM-algorithm
        startp = NULL,
        EMmaxit = 500,
        EMdev.change = 0.001,
        pr.it = FALSE
        )
```

`pattnpml.fit()` is a wrapper function for `alldistPC()`
which in turn is a modification of `alldist()` from the npmlreg package (Einbeck, Darnell, and Hinde, 2007)
modification allows for multiple random effect terms
more flexibility in choosing starting values

NPML example: Sources of Science information

Eurobarometer 55.2 May-June 2001 Question 5.

Here are some sources of information about scientific developments. Please rank them from 1 to 6 in terms of their importance to you (1 being the most important and 6 the least important)

a) Television .....
b) Radio .....
c) Newspapers and magazines .....
d) Scientific magazines .....
e) The internet .....
f) School/University .....

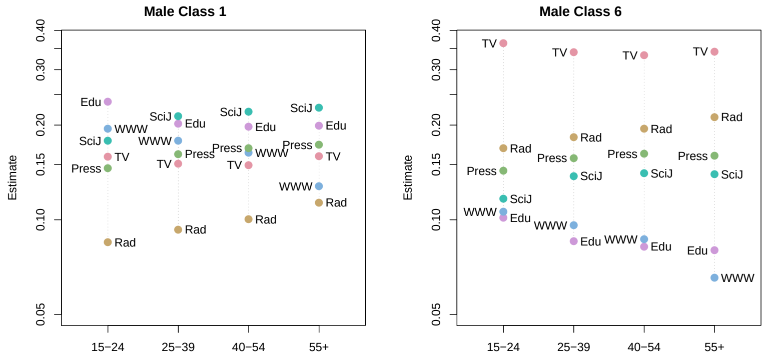
12216 complete rankings of the 6 objects: TV, Radio, . . .
subject covariates:
`AGE` (4 levels: 15-24, 25-39, 40-54 and 55+)
`SEX` (2 levels: male, female)

Example: Model selection

- find fitting fixed effects model: `AGE + SEX`
- fit `AGE + SEX` random effects model with increasing number of mass points
- each model was fitted 50 times with different starting values
- model with smalles BIC was selected ($*$)

	(a) without covariates			(b) with AGE and SEX		
No. of mass points r	Deviance	No. of para-meters	BIC	Deviance	No. of para-meters	BIC
1	21293	13	21406	17815	33	18100
2	12494	18	12650	10731	38	11060
3	10252	23	10451	9056	43	9428
4	9792	28	10035	8836	48	9252
5	9544	33	9830	8729	53	9187
6	9387	38	9716	8667	58	$*$ 9170
7	9302	43	9674	8636	63	9182
8	9277	48	9693	8623	68	9212

Results



Extension 2: Missing observations in paired comparisons

missing observations can occur for several reasons:
by design, respondent doesn't know, is unwilling, fatigue, etc.

if NA occurs at random – easily handled in LLBT
since $m_{(y_{jk})}$ depend only on observed values

but we want to use pattern models for several reasons

how can we take account of incomplete response patterns?

- each different missing pattern gives a different design matrix (smaller than design matrix for non-missing data)
- likelihood is computed for each of these "different" tables "individual" contributions to the likelihood
- total likelihood (which is then maximised) is the product of all the "individual" contributions

Data structure

	observed patterns			complete patterns			NA patterns		
	y_{12}	y_{13}	y_{23}	(12)	(13)	(23)	(12)	(13)	(23)
block 1 []	1	1	1	1	1	1	0	0	0
	1	1	−1	1	1	−1	0	0	0
	1	−1	1	1	−1	1	0	0	0
	1	−1	−1	1	−1	−1	0	0	0
	−1	1	1	−1	1	1	0	0	0
	−1	1	−1	−1	1	−1	0	0	0
	−1	−1	1	−1	−1	1	0	0	0
	−1	−1	−1	−1	−1	−1	0	0	0
block 2: [23]	1	1	NA	1	1	1	0	0	1
				1	1	−1	0	0	1
	1	−1	NA	1	−1	1	0	0	1
				1	−1	−1	0	0	1
	−1	1	NA	−1	1	1	0	0	1
				−1	1	−1	0	0	1
	−1	−1	NA	−1	−1	1	0	0	1
				−1	−1	−1	0	0	1
block 3		⋮			⋮		⋮		

- $P_{obs}(1,1,\text{NA}) = P_{compl}(1,1,1) + P_{compl}(1,1,-1)$

Modelling missing values

complete data is table with $2^{2\ell}$ cells
cell probability is $P\{Y = y, R = r;\ \pi, \psi\}$

NA model:

$$P\{Y = y, R = r;\ \pi, \psi\} = P\{Y = y;\ \pi\}P\{R = r|Y = y;\ \psi\} = f(y)q(r|y)$$

cell probabilities for incomplete (observed data):

$$\begin{aligned} P\{y_{12}, y_{13}, y_{23};\ \pi, \psi\} &= f(y_{12}, y_{13}, y_{23};\ \pi)\, q(0,0,0 \,|\, y_{12}, y_{13}, y_{23}; \psi) \\ P\{y_{12}, y_{13}, \text{NA};\ \pi, \psi\} &= \textstyle\sum_{y_{23}} f(y_{12}, y_{13}, y_{23};\ \pi)\, q(0,0,1 \,|\, y_{12}, y_{13}, y_{23}; \psi) \\ P\{y_{12}, \text{NA}, y_{23};\ \pi, \psi\} &= \textstyle\sum_{y_{13}} f(y_{12}, y_{13}, y_{23};\ \pi)\, q(0,1,0 \,|\, y_{12}, y_{13}, y_{23}; \psi) \\ &\ \vdots \end{aligned}$$

this is a composite link approach (Thompson & Baker, 1981):
extending GLMs: $\mu_i = c_i h(\gamma) = \sum c_{ik} h(\eta_k)$
c_i's are known functions (CL functions)

Missing data mechanisms (Rubin, 1976)

let $y = (y_{obs}, y_{mis})$ and R_{jk} be an NA indicator (if NA: $R_{jk} = 1$)

Missing completely at random (MCAR):
If the conditional distribution $P\{R = r \mid Y = y;\ \psi\}$ is independent of Y, i.e. $P\{R = r \mid Y = y;\ \psi\} = P\{R = r;\ \psi\}$.

Missing at random (MAR):
If the conditional distribution depends on the observed, but not on the missing values, $P\{R = r \mid Y = y;\ \psi\} = P\{R = r \mid Y_{obs} = y_{obs};\ \psi\}$.

Missing not at random (MNAR):
If the conditional distribution depends on both the observed and the missing values,
$P\{R = r \mid Y = y;\ \psi\} = P\{R = r \mid Y_{obs} = y_{obs}, Y_{mis} = y_{mis};\ \psi\}$.

Estimation of the outcome model $f(y)$

total likelihood is product of likelihoods for each NA pattern block [·]

$$L(\lambda; y) = L_{[\,]} \cdot L_{[12]} \cdots L_{[12][13]} \cdots L_{[12\ldots J]}$$

individual contributions are:

$$L_{[\,]} = \prod_{y \in Y_{[\,]}} P(y; \pi, \psi)^{n_y} = \prod_{y \in Y_{[\,]}} \left(\frac{\exp\{\eta_{(y_{12}, y_{13}, \ldots, y_{J-1,J})}\}}{\sum_{y \in Y_{[\,]}} \exp\{\eta_y\}} \right)^{n_y}$$

and, e.g.,

$$L_{[12]} = \prod_{y \in Y_{[12]}} \left(\frac{\exp\{\eta_{(1, y_{13}, \ldots, y_{J-1,J})}\} + \exp\{\eta_{(-1, y_{13}, \ldots, y_{J-1,J})}\}}{\sum_{y \in Y_{[\,]}} \exp\{\eta_y\}} \right)^{n_y}$$

Some nonresponse models: $q(r \mid y)$

- under MCAR assumption:

model 1: $P\{R_{jk} = r_{jk}\} = e^{\alpha_{jk} r_{jk}} / (1 + e^{\alpha_{jk}})$, $r_{jk} \in \{0, 1\}$
model 2: common α, i.e., $\alpha_{jk} = \alpha$
model 3: reparameterise α_{jk} with $\alpha_j + \alpha_k$

- under MNAR assumption: (include dependence on y)

model 1: $P\{R_{jk} = r_{jk} | Y_{jk} = y_{jk}\} = e^{(\alpha_{jk} + y_{jkj}\beta_{jk}) r_{jk}} / (1 + e^{\alpha_{jk} + y_{jk}\beta_{jk}})$
model 2: common α and β
model 3: additionally reparameterise β_{jk} with $\beta_j + \beta_k$

Estimation:

linear predictors of outcome model η_y are extended to $\eta_y + \eta_{r|y}$
apart from that, the procedure remains the same as for the pure outcome model

The missing observations model in prefmod

some nonresponse models for missing observations are handled using further arguments in the pattern model functions

```
pattPC.fit(obj, nitems, formel = ~1, elim = ~1, resptype = "paircomp",
    obj.names = NULL, undec = FALSE, ia = FALSE, NItest = FALSE,
    NI = FALSE, MIScommon = FALSE, MISmodel = "obj", MISalpha = NULL,
    MISbeta = NULL, pr.it = FALSE)
```

`NItest` ...separate estimation for complete and incomplete patterns
`NI` ...large table (crossclassification with NA patterns)
`MIScommon` ...fits a common parameter for NA indicators, i.e., $\alpha = \alpha_j = \alpha_k$
`MISalpha` ...specification to fit parameters for NA indicators using α_{ij} or $\alpha_i + \alpha_j$
`MISbeta` ...fits parameters for MNAR model, analogous to `MISalpha`

same arguments available for `pattR.fit()` and `pattL.fit()`

Missing values example: Attitudes towards foreigners

Survey at the Vienna University of Economics(Weber, 2010)

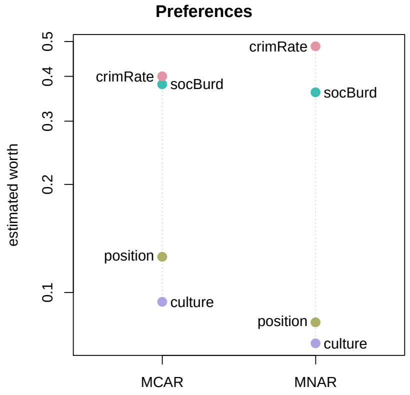
98 students rated four extreme statements about hypothetical consequences of migration through a paired comparison experiment

1)	crimRate	Foreigners increase crime rates
2)	position	Foreigners take away training positions
3)	socBurd	Foreigners are a burden for the social welfare system
4)	culture	Foreigners threaten our culture

```
> MCAR <- pattPC.fit(immig, 4, undec = T)

> MNAR <- pattPC.fit(immig, 4, undec = T, MISalpha = c(T, T, T, T),
+    MISbeta = c(T, T, T, T))
```

Example (cont'd)



Extension 3: Multivariate responses

– repeated observations of paired comparisons over time
– cross-sectional comparisons according to different attributes

formulation as pattern model straightforward
a response pattern is

$$\{y_{121}, \ldots, y_{12T}, \ldots, y_{jk1}, \ldots y_{jkT}, \ldots, y_{(J-1)J1}, \ldots, y_{(J-1)JT}\}$$

however pattern model intractable:
e.g., 5 items at 3 time points results in 2^{30} patterns

idea: combination of LLBT and pattern model assuming:
– independence between comparisons (LLBT)
– patterns within comparisons (time points)

Multivariate LLBT

extending the LLBT we get

$$\ln m_{(jk)(y_{jk1} \cdots y_{jkT})} = \mu_{(jk)} + \sum_{t=1}^{T} y_{jkt}(\lambda_{jt} - \lambda_{kt}) + \sum_{s<t} y_{jks} y_{jkt} \zeta_{(jk)(st)}$$

for 2 time points and for a certain comparison (jk)

$$\begin{aligned}
\ln m_{(jk)(++)} &= \mu_{(jk)} + \lambda_{j1} - \lambda_{k1} + \lambda_{j2} - \lambda_{k2} + \zeta_{(jk)} \\
\ln m_{(jk)(-+)} &= \mu_{(jk)} - \lambda_{j1} + \lambda_{k1} + \lambda_{j2} - \lambda_{k2} - \zeta_{(jk)} \\
\ln m_{(jk)(+-)} &= \mu_{(jk)} + \lambda_{j1} - \lambda_{k1} - \lambda_{j2} + \lambda_{k2} - \zeta_{(jk)} \\
\ln m_{(jk)(--)} &= \mu_{(jk)} - \lambda_{j1} + \lambda_{k1} - \lambda_{j2} + \lambda_{k2} + \zeta_{(jk)}
\end{aligned}$$

Within-comparison dependence

for 2 time points there are $\binom{J}{2}$ *within-comparison* dependencies
for T time points there are $\binom{T}{2} \times \binom{J}{2}$ such dependencies

interpretation of $\zeta_{(jk)(st)}$

			time 2 (1 > 2)	time 2 (2 > 1)
			+	−
time 1	(1 > 2)	+	m_{++}	m_{+-}
	(2 > 1)	−	m_{-+}	m_{--}

$$\ln OR_{(jk)} = \ln \frac{m_{++} m_{--}}{m_{+-} m_{-+}} = 4\zeta_{(jk)}$$

restrictions on $\zeta_{(jk)(st)}$ allow for modelling the association structure

Modelling change

specifying a design matrix $\boldsymbol{W}$ for the objects allows for a reparameterisation reflecting certain "change"-hypotheses

e.g., 3 objects 2 time points, $\delta_j = \lambda_{j2} - \lambda_{j1}$

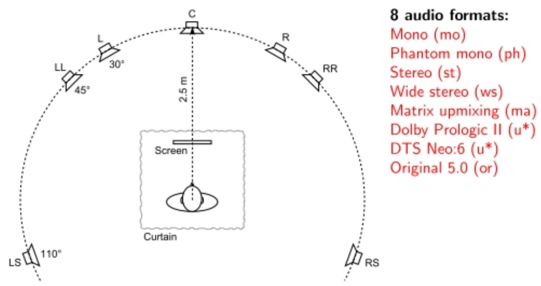
$$\boldsymbol{W} = \begin{array}{c} \\ \lambda_{11} \\ \lambda_{21} \\ \lambda_{31} \\ \lambda_{12} \\ \lambda_{22} \\ \lambda_{32} \end{array} \begin{array}{c} \begin{array}{cccccc} \lambda_{11} & \lambda_{21} & \lambda_{31} & \delta_1 & \delta_2 & \delta_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

other choices of $\boldsymbol{W}$ allow for different hypotheses, e.g., $\delta_1 = \delta_2$

Example: Psychacoustics

Application: Perceptual evaluation of multichannel sound

(Choisel & Wickelmaier, 2006, JAES)

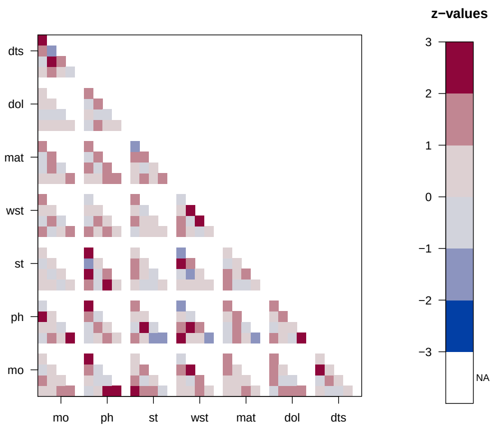


for details ask Florian ☺
we fit a model with 8 objects and 5 timepoints

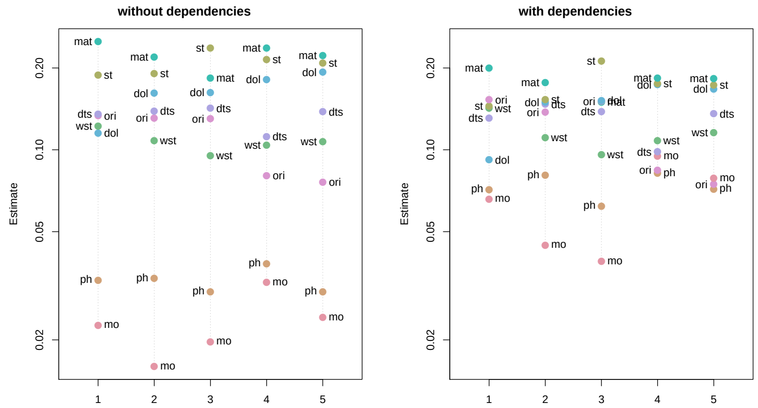
Example (cont'd): association structure

time points

12
13 23
14 24 34
15 25 35 45



Example (cont'd): worth plots



Some References

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