Funded by

Deutsche Forschungsgemeinschaft

German Research Foundation

technische universität dortmund

Fisher-z based Confidence Intervals of Correlations in Fixed- and Random-Effects Meta-Analysis

Psychoco 2020

Thilo Welz

Faculty of Statistics, TU Dortmund University

February 27-28, 2020

Motivation

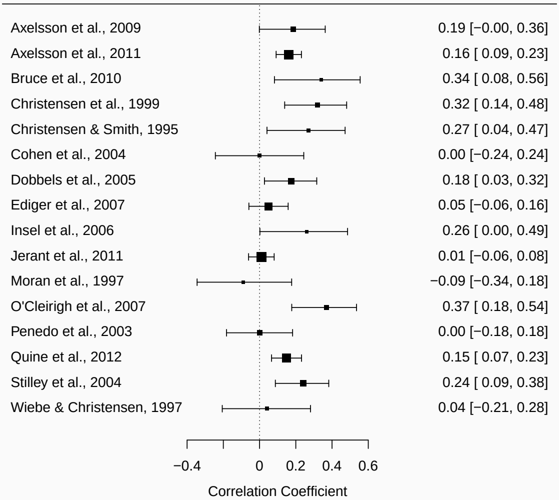
Data Example

	authors	year	n_i	r_i	v_i
1	Axelsson et al.	2009	109	0.19	0.01
2	Axelsson et al.	2011	749	0.16	0.00
3	Bruce et al.	2010	55	0.34	0.01
4	Christensen et al.	1999	107	0.32	0.01
5	Christensen & Smith	1995	72	0.27	0.01
6	Cohen et al.	2004	65	0.00	0.02
7	Dobbels et al.	2005	174	0.17	0.01
8	Ediger et al.	2007	326	0.05	0.00
9	Insel et al.	2006	58	0.26	0.02
10	Jerant et al.	2011	771	0.01	0.00
11	Moran et al.	1997	56	-0.09	0.02
12	O'Cleirigh et al.	2007	91	0.37	0.01
13	Penedo et al.	2003	116	0.00	0.01
14	Quine et al.	2012	537	0.15	0.00
15	Stilley et al.	2004	158	0.24	0.01
16	Wiebe & Christensen	1997	65	0.04	0.02

Table 1: Meta-Analysis of 16 studies on the correlation between medication adherence and conscientiousness; n_i : sample size, r_i: study effect, v_i : Variance estimate of effect

Visualization via forestplots

Study results with two-sided confidence intervals



The random-effects meta-analysis (REMA) model

The REMA model for K studies:

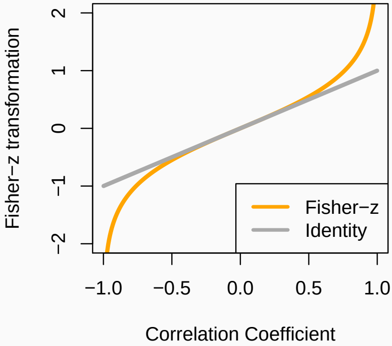
$$\mu_i = \mu + u_i + \varepsilon_i, \ \ i = 1, \ldots, K$$

with $u_i \sim N(0, \tau^2)$ and $\varepsilon_i \sim N(0, \sigma_i^2)$.

- The main effect μ is estimated as a weighted average, using inverse variance weights, i.e. $\hat{\mu} = \frac{\sum_{i=1}^K w_i \mu_i}{\sum_{i=1}^K w_i}$ with $w_i = (\hat{\sigma}_i^2 + \hat{\tau}^2)^{-1}$.
- Multiple estimators for τ^2 exist. Common choices are REML and DerSimonian-Laird.

The Fisher-z transformation

For bivariate (X_i, Y_i), $i = 1, \dots, N$, with sample correlation coefficient $\hat{\varrho}$ the Fisher-z transformation of $\hat{\varrho}$ is $z = \frac{1}{2}\ln(\frac{1+\hat{\varrho}}{1-\hat{\varrho}}) = \text{arctanh}(\hat{\varrho})$.



If (X, Y) is normally distributed with correlation ϱ and the $(X_i, Y_i)_{i=1}^{N}$ are iid, then $z \underset{approx}{\overset{d}{\sim}} N(\frac{1}{2}\ln(\frac{1+\varrho}{1-\varrho}), \frac{1}{N-3})$.

Confidence Intervals for ϱ

Previously suggested confidence intervals

- HOVz (Hedges-Olkin-Vevea Fisher-z) Approach[1]:

$$\tanh\left(\hat{z} \pm u_{1-\alpha/2} \cdot w^{-1/2}\right),$$

with $\hat{z} = \sum_i \frac{w_i}{w}\hat{z}_i$, $w = \sum_{i=1}^K w_i$, $w_i = (\frac{1}{n_i-3} + \hat{\tau}^2)^{-1}$ and $u_{1-\alpha/2}$ the $(1-\alpha/2)$-quantile of the standard normal distribution.

- HS (Hunter-Schmidt)[2]:

$$\hat{\varrho} \pm u_{1-\alpha/2} \cdot \sqrt{\widehat{Var}(\hat{\varrho})},$$

where $\hat{\varrho} = \frac{\sum_i n_i \hat{\varrho}_i}{\sum_i n_i}$ and $\widehat{Var}(\hat{\varrho}) = \frac{\sum_i n_i(\hat{\varrho}_i - \hat{\varrho})^2}{K \cdot \sum_i n_i}$.

Problem: Both perform poorly in simulations (especially HS!)

[1] Hafdahl and Williams (2009)
[2] Schulze (2004)

Our new suggestions I

Our suggested improvements to HOVz: Better estimators for $Var(\hat{z})$ and $(t_{K-1,1-\alpha/2})$-quantiles instead of the standard normal quantiles $u_{1-\alpha/2}$:

Knapp-Hartung[3]:

$$\tanh\left(\hat{z} \pm t_{K-1,1-\alpha/2} \cdot \sqrt{\widehat{Var}_{KH}(\hat{z})}\right),$$

with $\widehat{Var}_{KH}(\hat{z}) = \frac{1}{K-1}\sum_{i=1}^{K} \frac{w_i}{w}(\hat{z}_i - \hat{z})^2$ and $w = \sum_{i=1}^{K} w_i$.

[3] Hartung (1999)

Our new suggestions II

HC-estimators[4]:

$$\tanh\left(\hat{z} \pm t_{K-1,1-\alpha/2} \cdot \sqrt{\widehat{Var}_{HC}(\hat{z})}\right),$$

with $\widehat{Var}_{HC}(\hat{z}) = \frac{1}{(\sum_i w_i)^2} \sum_{i=1}^{K} w_i^2 \hat{\varepsilon}_i^2 (1 - x_{ii})^{-\delta_i}$, $x_{ii} = \frac{w_i}{\sum_j w_j}$, $\hat{\varepsilon}_i = \hat{z}_i - \hat{z}$.
δ_i depends on choice of HC estimator.

$$HC_3: \ \delta_i = 2$$

$$HC_4: \ \delta_i = \min\left\{4, \frac{x_{ii}}{\bar{x}}\right\} = \min\{4, K \cdot x_{ii}\}$$

[4] Welz and Pauly (2020); Cribari-Neto et al. (2007)

Our new suggestions III

- Wild Bootstrap variance estimators[5]:

$$\tanh\left(\hat{z} \pm t_{K-1,1-\alpha/2} \cdot \sqrt{\widehat{Var}^*(\hat{z})}\right),$$

  $\widehat{Var}^*(\hat{z})$ is the empirical variance of $\hat{z}^*_b = \frac{\sum_i w^*_{ib}\hat{z}^*_{ib}}{\sum_i w^*_{ib}}$, with $w^*_{ib} = (\frac{1}{n_i-3} + \hat{\tau}^{*2}_b)^{-1},\ b = 1, \ldots, B.$
- Study level bootstrap estimates are generated via $\hat{z}^*_{ib} = \hat{z}_i + \hat{\varepsilon}_i \cdot \nu_i$ with $\nu_i \overset{d}{\sim} N(0, \gamma)$ and $\gamma \in \left\{1, \frac{K-1}{K-3}, \frac{K-2}{K-3}\right\}$. The latter choices require $K \geq 4$ studies.

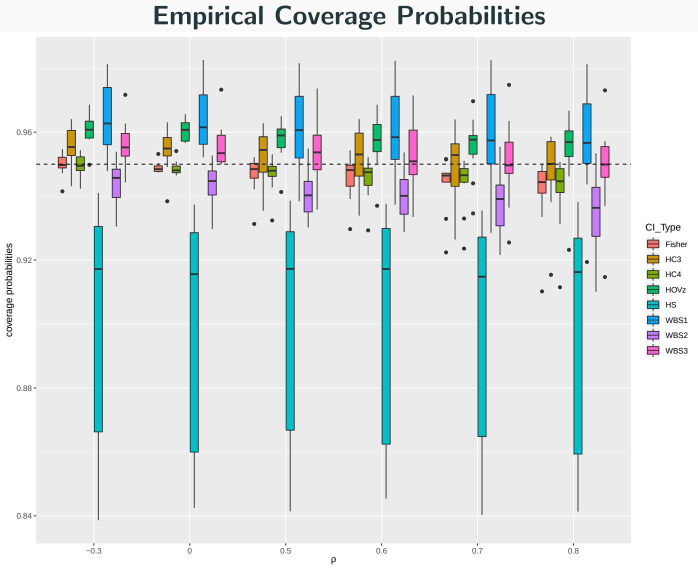
[5] Davidson and Flachaire (2008)

Simulation & Results

The Simulation Setup

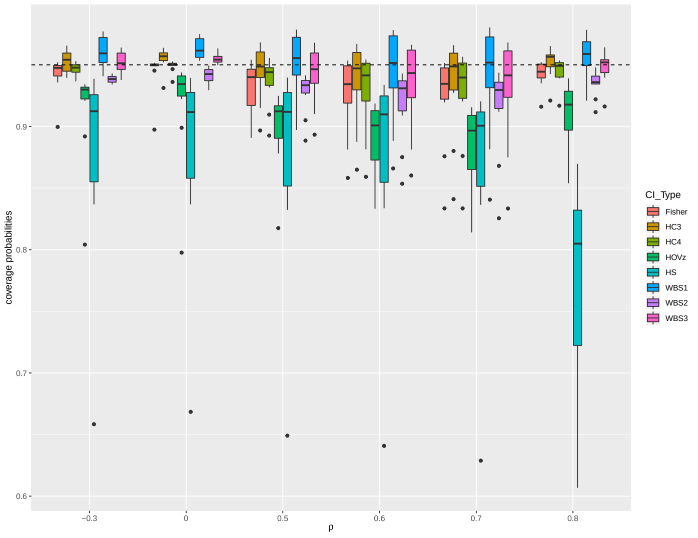
- Model: $\varrho_i = \varrho + u_i + \varepsilon_i$ with
  $\varrho_i \sim TN(\varrho, \sigma_i^2 + \tau^2, \min = -.999, \max = .999)$
- Effect: Pearson's Correlation coefficient
- Simulation parameters:
  - Number of studies: $K \in \{5, 10, 20, 40\}$
  - Between-study variance τ^2 (heterogeneity): $\tau \in \{0, 0.16, 0.4\}$
  - Varying number of patients per study i: $15 \leq n_i \leq 400$
  - $N = 10000$ simulation runs, $\alpha = 0.05$

Results $\tau = 0$

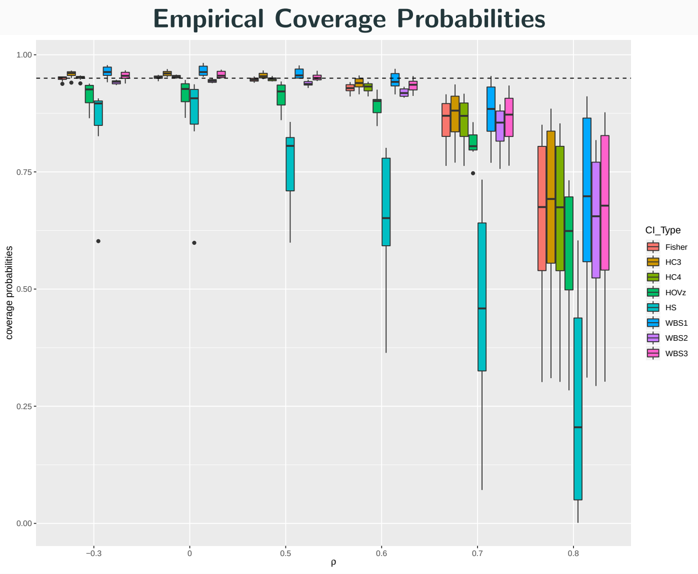


Results $\tau = 0.16$

Empirical Coverage Probabilities



Results $\tau = 0.4$



Summary

Summary

- HS performs terribly (even in the FE model!)
- HOVz also with incorrect coverage (especially for $\tau >> 0$)
- New CI's with most accurate coverage (all based on Fisher-z transformation):
  1. Knapp-Hartung method (Fisher)
  2. Robust variance estimator (HC_4)
  3. Data dependent wild bootstrap approach (WBS3)

However:

The coverage of considered CI's is still poor for $\tau >> 0$ and $|\varrho|$ close to 1.

References

Cribari-Neto, F., Souza, T. C., and Vasconcellos, K. L. (2007). Inference under heteroskedasticity and leveraged data. *Communications in StatisticsTheory and Methods*, 36(10):1877–1888.

Davidson, R. and Flachaire, E. (2008). The wild bootstrap, tamed at last. *Journal of Econometrics*, 146(1):162–169.

Hafdahl, A. R. and Williams, M. A. (2009). Meta-analysis of correlations revisited: Attempted replication and extension of field's (2001) simulation studies. *Psychological Methods*, 14(1):24.

Hartung, J. (1999). An alternative method for meta-analysis. *Biometrical Journal: Journal of Mathematical Methods in Biosciences*, 41(8):901–916.

Schulze, R. (2004). *Meta-analysis-A comparison of approaches*. Hogrefe Publishing.

Welz, T. and Pauly, M. (2020). A simulation study to compare robust tests for linear mixed-effects meta-regression. *Research Synthesis Methods*.

Thank you
for your attention!